

WEAK HOPF MONOIDS IN BRAIDED MONOIDAL CATEGORIES

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ABSTRACT. We develop the theory of weak bimonoids in braided monoidal categories and show them to be quantum categories in a certain sense. Weak Hopf monoids are shown to be quantum groupoids. Each separable Frobenius monoid R leads to a weak Hopf monoid $R \otimes R$.

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1. INTRODUCTION

Weak Hopf algebras were introduced by Böhm, Nill, and Szlachányi in a series of papers [5, 15, 22, 4]. They are generalizations of Hopf algebras and were proposed as an alternative to weak quasi-Hopf algebras. A weak bialgebra is both an associative algebra and a coassociative coalgebra, but instead of requiring that the multiplication and unit morphism are coalgebra morphisms (or equivalently that the comultiplication and the counit are algebra morphisms) other “weakened” axioms are imposed. The multiplication is still required to be comultiplicative (equivalently, the comultiplication is still required to be multiplicative), but the counit is no longer required to be an algebra morphism and the unit is no longer required to be a coalgebra morphism. Instead, these requirements are replaced by weakened versions (see equations (v) and (w) below). As the name suggests, any bialgebra satisfies these weakened axioms and is therefore a weak bialgebra.

Given a weak bialgebra A one may define source and target morphisms $s, t : A \longrightarrow A$ whose images $s(A)$ and $t(A)$ are called the “source and target (counital)

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subalgebras". It has been shown by Nill [15] that Hayashi's face algebras [11] are special cases of weak bialgebras for which the, say, target subalgebra is commutative.

A weak Hopf algebra is a weak bialgebra H equipped with an antipode $\nu : H \longrightarrow H$ satisfying the axioms¹

$$\mu(\nu \otimes 1)\delta = t, \quad \mu(1 \otimes \nu)\delta = s, \quad \text{and} \quad \mu_3(\nu \otimes 1 \otimes \nu)\delta_3 = \nu,$$

where $\mu_3 = \mu(\mu \otimes 1)$ and $\delta_3 = (\delta \otimes 1)\delta$. Again, any Hopf algebra satisfies these weakened axioms and so is a weak Hopf algebra. Also in [15] Nill has shown that the (finite dimensional) generalized Kac algebras of Yamanouchi [25] are examples of weak Hopf algebras with involutive antipode. Weak Hopf algebras have also been called "quantum groupoids" [16] and in this paper this is *not* what we mean by quantum groupoid.

Perhaps the simplest example of weak bialgebras and weak Hopf algebras are, respectively, category algebras and groupoid algebras. Suppose that k is a field and let \mathcal{C} be a category with set of object \mathcal{C}_0 and set of morphism \mathcal{C}_1 . The *category algebra* $k[\mathcal{C}]$ is the vector space $k[\mathcal{C}_1]$ over k with basis \mathcal{C}_1 . Elements are formal linear combinations of the elements of \mathcal{C}_1 with coefficients in k , i.e.,

$$\alpha f + \beta g + \dots$$

with $\alpha, \beta \in k$ and $f, g \in \mathcal{C}_1$. An associative multiplication on $k[\mathcal{C}]$ is defined by

$$\mu(f, g) = f \cdot g = \begin{cases} g \circ f & \text{if } g \circ f \text{ exists} \\ 0 & \text{otherwise} \end{cases}$$

and extended by linearity to $k[\mathcal{C}]$. This algebra does not have a unit unless \mathcal{C}_0 is finite, in which case the unit is

$$\eta(1) = e = \sum_{A \in \text{ob } \mathcal{C}} 1_A,$$

making $k[\mathcal{C}]$ into a unital algebra; all algebras (monoids) considered in this paper will be unital. A comultiplication and counit may be defined on $k[\mathcal{C}]$ as

$$\begin{aligned} \delta(f) &= f \otimes f \\ \epsilon(f) &= 1 \end{aligned}$$

making $k[\mathcal{C}]$ into a coalgebra. Note that $k[\mathcal{C}]$ equipped with this algebra and coalgebra structure will not satisfy any of the following usual bialgebra axioms:

$$\epsilon\mu = \epsilon \otimes \epsilon \quad \delta\eta = \eta \otimes \eta \quad \epsilon\eta = 1_k.$$

The one bialgebra axiom that does hold is $\delta\mu = (\mu \otimes \mu)(1 \otimes c \otimes 1)(\delta \otimes \delta)$. Equipped with this algebra and coalgebra structure $k[\mathcal{C}]$ does, however, satisfy the axioms of a weak bialgebra. Furthermore, if \mathcal{C} is a groupoid, then $k[\mathcal{C}]$, which is then called the *groupoid algebra*, is an example of a weak Hopf algebra with antipode $\nu : k[\mathcal{C}] \longrightarrow k[\mathcal{C}]$ defined by

$$\nu(f) = f^{-1}.$$

¹There may be some discrepancy with what we call the source and target morphisms and what exists in the literature. This arises from our convention of taking multiplication in the groupoid algebra to be $f \cdot g = g \circ f$ (whenever $g \circ f$ is defined).

and extended by linearity. If $f : A \longrightarrow B \in \mathcal{C}$, the source and target morphisms $s, t : k[\mathcal{C}] \longrightarrow k[\mathcal{C}]$ are given by

$$s(f) = 1_A \quad \text{and} \quad t(f) = 1_B,$$

as one would expect.

In this paper we define weak bialgebras and weak Hopf algebras in a braided monoidal category \mathcal{V} , where prefer to call them “weak bimonoids” and “weak Hopf monoids”. To define a weak bimonoid in \mathcal{V} the only difference from the definition given by Böhm, Nill, and Szlachányi [4] is that a choice of “crossing” must be made in the axioms. Our definition is not as general as the one given by J. N. Alonso Álvarez, J. M. Fernández Vilaboa, and R. González Rodríguez in [1, 2], but, in the case that their weak Yang-Baxter operator $t_{A,A}$ is the braiding $c_{A,A}$ and their idempotent $\nabla_{A \otimes A} = 1_{A \otimes A}$, then our choices of crossings are the same. Our difference in defining weak bimonoids occurs in the choice of source and target morphisms. We have chosen $s : A \longrightarrow A$ and $t : A \longrightarrow A$ so that:

- (1) the “globular” identities $ts = s$ and $st = t$ hold;
- (2) the source subcomonoid and target subcomonoid coincide (up to isomorphism), and is denoted by C ;
- (3) $s : A \longrightarrow C^\circ$ and $t : A \longrightarrow C$ are comonoid morphisms.

These properties of the source and target morphisms are essential for our view of quantum categories. These are $s = \bar{\Pi}_A^L$ and $t = \Pi_A^R$ in the notation of [1, 2] and $s = \epsilon'_s$ and $t = \epsilon_s$ in the notation of [19], with the appropriate choice of crossings.

We choose to work in the Cauchy completion \mathcal{QV} of \mathcal{V} . The category \mathcal{QV} is also called the “completion under idempotents” of \mathcal{V} or the “Karoubi envelope” of \mathcal{V} . This is done rather than assume that idempotents split in \mathcal{V} . Suppose that A is a weak bimonoid in \mathcal{QV} . In this case we find C by splitting either the source or target morphism. As in [19, Prop. 4.2], C is a separable Frobenius monoid in \mathcal{QV} , meaning that $(C, \mu, \eta, \delta, \epsilon)$ is a Frobenius monoid with $\mu\delta = 1_C$.

It turns out that our definition of weak Hopf monoid is (in the symmetric case) the same as what is proposed in [4], and in the braided case in [1, 2]. A weak bimonoid H is a weak Hopf monoid if it is equipped with an antipode $\nu : H \longrightarrow H$ satisfying

$$\mu(\nu \otimes 1)\delta = t, \quad \mu(1 \otimes \nu)\delta = r, \quad \text{and} \quad \mu_3(\nu \otimes 1 \otimes \nu)\delta_3 = \nu,$$

where $r = \nu s$. This $r : H \longrightarrow H$ here turns out to be the “usual” source morphism; Π_H^L in the notation of [1, 2]. Ignoring crossings r is ϵ_t in the notation of [19] and our r and t correspond respectively to \square^L and \square^R in the notation of [4]; the morphism s does not appear in [4]. Usually, in the second axiom above, $\mu(1 \otimes \nu)\delta = r$, the right-hand side is equal to the chosen source map s of the weak bimonoid H . The reason that this r does not work as a source morphism for us is that it does not satisfy all three requirements for the source morphism mentioned above. This choice of r allows us to show that any Frobenius monoid in \mathcal{V} yields a weak Hopf monoid $R \otimes R$ with bijective antipode (cf. the example in the Appendix of [4]).

There are a number of generalizations of bialgebras and Hopf algebras to their “many object” versions. For example, Sweedler’s generalized bialgebras [21], which were later generalized by Takeuchi to \times_R -bialgebras [23], the quantum groupoids of Lu [14] and Xu [24], Schauenburg’s \times_R -Hopf algebras [18], the bialgebroids and Hopf algebroids of Böhm and Szlachányi [7], the earlier mentioned face algebras [11]

and generalized Kac algebras [25], and, the ones of interest in this paper, the quantum categories and quantum groupoids of Day and Street [9]. It has been shown by Brzeziński and Militaru that the quantum groupoids of Lu and Xu are equivalent to Takeuchi's \times_R -bialgebras [8, Thm. 3.1]. Schauenburg has shown in [17] that face algebras are an example of \times_R -bialgebras for which R is commutative and separable. In [19, Thm. 5.1] Schauenburg has shown that weak bialgebras are also examples of \times_R -bialgebras for which R is separable Frobenius (there called Frobenius-separable). Schauenburg also shows in [19, Thm. 6.1] that a weak Hopf algebra may be characterized as a weak bialgebra H for which a certain canonical map $H \otimes_C H \longrightarrow \mu(\delta(\eta(1)), H \otimes H)$ is a bijection. As a corollary he shows that a weak Hopf algebra is a \times_R -Hopf algebra.

Quantum groupoids were introduced in [9]. They first introduce quantum categories. A quantum category in \mathcal{V} consists of two comonoids A and C in \mathcal{V} , with A playing the role of the object-of-morphisms and C the object-of-objects. There are source and target morphisms $s, t : A \longrightarrow C$, a “composition” morphism $\mu : A \otimes_C A \longrightarrow A$, and a “unit” morphism $\eta : C \longrightarrow A$ all in \mathcal{V} . This data must satisfy a number of axioms. Indeed, ordinary categories are examples of quantum categories. Motivated by the duality found in $*$ -autonomous categories [3], they then define a quantum groupoid to be a quantum category equipped with a generalized antipode coming from a $*$ -autonomous structure.

In this paper we show that weak bimonoids are examples of quantum categories for which the object-of-objects C is a separable Frobenius monoid, and that weak Hopf monoids with invertible antipode are quantum groupoids.

An outline of this paper is as follows:

In §2 we provide the definition of weak bimonoid A in a braided monoidal category \mathcal{V} and define the source and target morphisms. We then move to the Cauchy completion $\mathcal{Q}\mathcal{V}$ and prove the three required properties of our source and target morphisms mentioned above. In this section we also prove that C , the object-of-objects of A , is a separable Frobenius monoid.

Weak Hopf monoids in braided monoidal categories are introduced in §3.

In §4 we describe a monoidal structure on the categories $\mathbf{Bicomod}(C)$ of C -bicomodules in \mathcal{V} , and $\mathbf{Comod}(A)$ of right A -comodules in \mathcal{V} , such that the underlying functor

$$U : \mathbf{Comod}(A) \longrightarrow \mathbf{Bicomod}(C)$$

is strong monoidal. If H is a weak Hopf monoid, then we are able to show that the category $\mathbf{Comod}_f(H)$, consisting of the dualizable objects of $\mathbf{Comod}(H)$, is left autonomous.

In §5 we prove that any separable Frobenius monoid R in a braided monoidal category \mathcal{V} yields an example of a weak Hopf monoid $R \otimes R$ with invertible antipode in \mathcal{V} .

The definitions of quantum categories and quantum groupoids are recalled in §6, and in §7 we show that any weak bimonoid is a quantum category and any weak Hopf monoid with invertible antipode is a quantum groupoid.

This paper depends heavily on of the string diagrams in braided monoidal categories of Joyal and Street [13], which were shown to be rigorous in [12]. The reader unfamiliar with string diagrams may first want to read Appendix A where we review some preliminary concepts using these diagrams.

We would like to thank J. N. Alonso Álvarez, J. M. Fernández Vilaboa, and R. González Rodríguez for sending us copies of their preprints [1, 2].

2. WEAK BIMONOIDS

A weak bialgebra [5, 15, 22, 4] is a generalization of a bialgebra with weakened axioms. These weakened axioms replace the three axioms that say that the unit is a coalgebra morphism and the counit is an algebra morphism. With the appropriate choices of under and over crossings the definition of a weak bialgebra carries over rather straightforwardly into braided monoidal categories, where we prefer to call it a “weak bimonoid”.

2.1. Weak bimonoids. Suppose that $\mathcal{V} = (\mathcal{V}, \otimes, I, c)$ is a braided monoidal category.

Definition 2.1. A *weak bimonoid* $A = (A, \mu, \eta, \delta, \epsilon)$ in \mathcal{V} is an object $A \in \mathcal{V}$ equipped with the structure of a monoid (A, μ, η) and a comonoid (A, δ, ϵ) satisfying the following equations.

$$\begin{aligned}
 \text{(b)} \quad & \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} | \quad | \\ \diagdown \quad \diagup \end{array} \\
 \text{(v)} \quad & \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \end{array} = \begin{array}{c} | \quad | \quad | \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \end{array} \\
 \text{(w)} \quad & \begin{array}{c} | \quad | \quad | \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} | \quad | \quad | \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \end{array}
 \end{aligned}$$

Suppose A and B are weak bimonoids in \mathcal{V} . A *morphism of weak bimonoids* $f : A \longrightarrow B$ is a morphism $f : A \longrightarrow B$ in \mathcal{V} which is both a monoid morphism and a comonoid morphism.

Let A be a weak bimonoid and define the *source* and *target* morphisms $s, t : A \longrightarrow A$ of A as follows:

$$s = \begin{array}{c} \circ \quad \circ \\ | \quad | \\ \diagdown \quad \diagup \end{array} \quad t = \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \circ \quad \circ \end{array}.$$

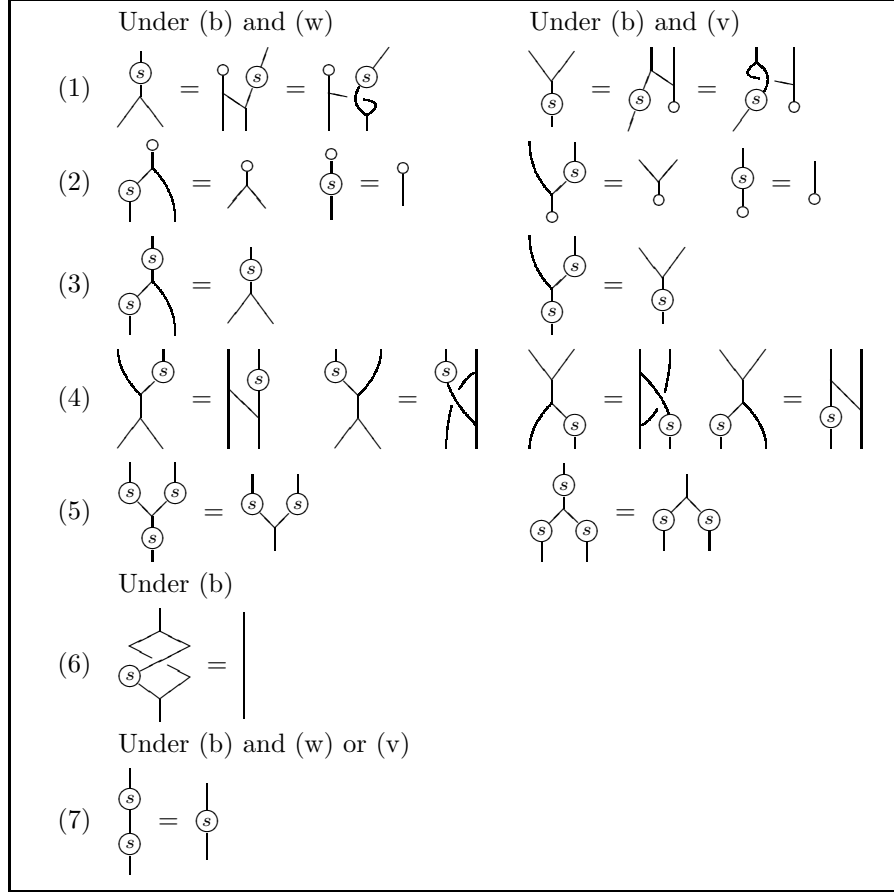
Notice that $s : A \longrightarrow A$ is invariant under rotation by π , while $t : A \longrightarrow A$ is invariant under horizontal reflection and the inverse braiding. Importantly, under either of these transformations

- (m) and (c) are interchanged²,
- (b) is invariant, and
- (v) and (w) are interchanged.

Note that these are not the “usual” source and target morphisms. They were chosen, as mentioned in the introduction, precisely because of the need for them to satisfy the following three properties:

- (1) the “globular” identities $ts = s$ and $st = t$ hold;
- (2) the source subcomonoid and target subcomonoid coincide (up to isomorphism), and is denoted by C ;

²The (m) and (c) here refer to the monoid and comonoid identities found in Appendix A.

FIGURE 1. Properties of s

(3) $s : A \longrightarrow C^\circ$ and $t : A \longrightarrow C$ are comonoid morphisms.

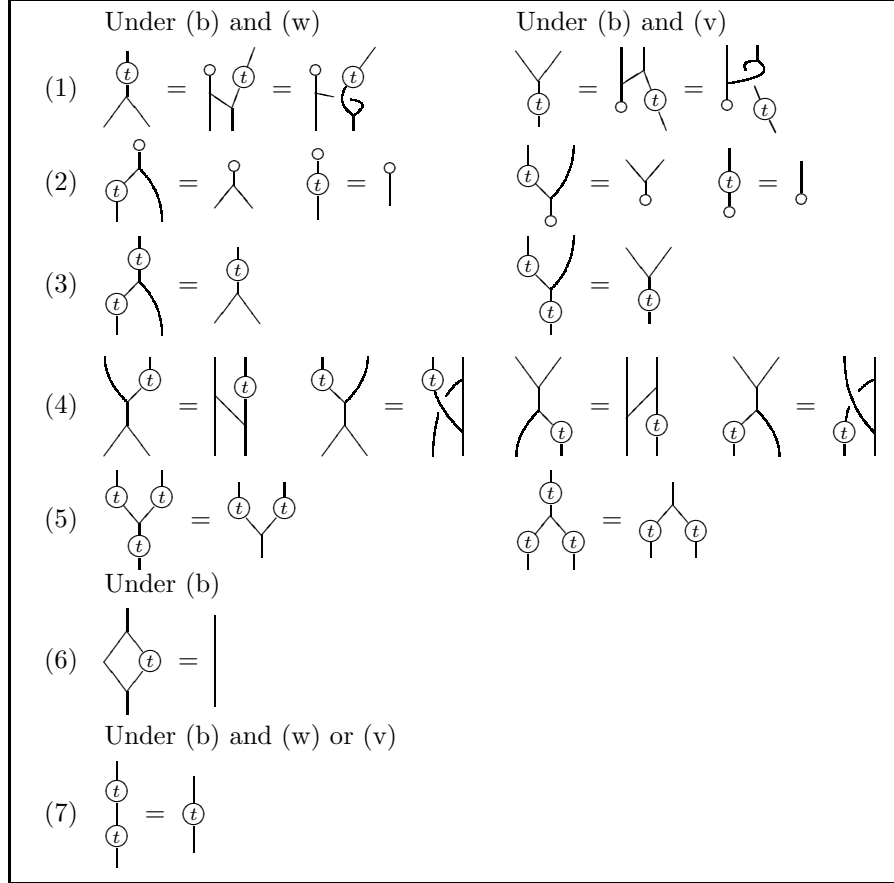
These properties will be proved in this section. Note that we will run into the usual source morphism (which we call r) in the definition of weak Hopf monoids (Definition 3.1).

A table of properties of the source morphism s is given in Figure 1 and table of properties of the target morphism t in Figure 2. Properties involving the interaction of s and t are given in Figure 3. Proofs of these properties may be found in Appendix B.

In the sequel $A = (A, \mu, \eta, \delta, \epsilon)$ will always denote a weak bimonoid and $s, t : A \longrightarrow A$ the source and target morphisms.

We see from property (7) in Figures 1 and 2 respectively that both s and t are idempotents. In the following we will work in the Cauchy completion (= completion under idempotents = Karoubi envelope) $\mathcal{Q}\mathcal{V}$ of \mathcal{V} . We do this rather than assume that idempotents split in \mathcal{V} .

2.2. Cauchy completion. Given a category \mathcal{V} , its *Cauchy completion* $\mathcal{Q}\mathcal{V}$ is the category whose objects are pairs (X, e) with $X \in \mathcal{V}$ and $e : X \longrightarrow X \in \mathcal{V}$ an

FIGURE 2. Properties of t

idempotent. A morphism $(X, e) \longrightarrow (X', e')$ in $\mathcal{Q}\mathcal{V}$ is a morphism $f : X \longrightarrow X' \in \mathcal{V}$ such that $e'fe = f$. Note that the identity morphism of (X, e) is e itself.

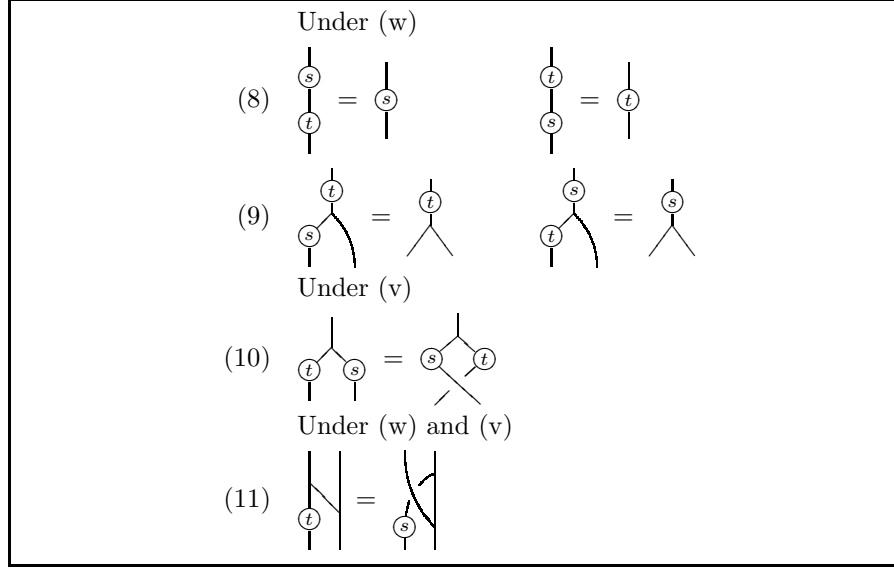
The point of working in the Cauchy completion is that every idempotent $f : (X, e) \longrightarrow (X, e)$ in $\mathcal{Q}\mathcal{V}$ has a splitting, viz.,

$$\begin{array}{ccc} (X, e) & \xrightarrow{f} & (X, e) \\ & \searrow f \quad \nearrow f & \\ & (X, f) & \end{array}$$

If \mathcal{V} is a monoidal category, then $\mathcal{Q}\mathcal{V}$ is a monoidal category via

$$(X, e) \otimes (X', e') = (X \otimes X', e \otimes e').$$

The category \mathcal{V} may be fully embedded in $\mathcal{Q}\mathcal{V}$ by sending $X \in \mathcal{V}$ to $(X, 1) \in \mathcal{Q}\mathcal{V}$ and $f : X \longrightarrow Y \in \mathcal{V}$ to $f : (X, 1) \longrightarrow (Y, 1)$, which is obviously a morphism in $\mathcal{Q}\mathcal{V}$. When working in $\mathcal{Q}\mathcal{V}$ we will often identify an object $X \in \mathcal{V}$ with $(X, 1) \in \mathcal{Q}\mathcal{V}$.

FIGURE 3. Interactions of s and t

2.3. Properties of the source and target morphisms. Let $A = (A, 1)$ be a weak bimonoid in \mathcal{QV} . From the definition of the Cauchy completion the result of splitting the source morphism s is (A, s) , and similarly, the result of splitting the target morphism t is (A, t) . The following proposition shows that these two objects are isomorphic.

Proposition 2.2. *The idempotent $t : (A, 1) \longrightarrow (A, 1)$ has the following two splittings.*

$$\begin{array}{ccc}
 (A, 1) & \xrightarrow{t} & (A, 1) \\
 & \searrow t & \nearrow t \\
 & (A, t) &
 \end{array}
 \qquad
 \begin{array}{ccc}
 (A, 1) & \xrightarrow{t} & (A, 1) \\
 & \searrow t & \nearrow s \\
 & (A, s) &
 \end{array}$$

In this case $s : (A, s) \longrightarrow (A, t)$ and $t : (A, t) \longrightarrow (A, s)$ are inverse morphisms, and hence $(A, t) \cong (A, s)$.

Proof. This result follows from the identities $ts = s$ and $st = t$ (property (8) in Figure 3). \square

We will denote this object by $C = (A, t)$ and call it the *object-of-objects* of A . In the next propositions we will show that C is a comonoid, and furthermore, that it is a separable Frobenius monoid, similar to what was done in [19] (there called Frobenius-separable).

Proposition 2.3. *The object $C = (A, t)$ equipped with*

$$\begin{aligned}
 \delta &= (C \xrightarrow{\delta} C \otimes C \xrightarrow{t \otimes t} C \otimes C) \\
 \epsilon &= C \xrightarrow{\epsilon} I
 \end{aligned}$$

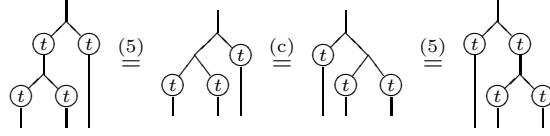
is a comonoid in \mathcal{QV} , and if furthermore equipped with

$$\begin{aligned}\mu &= (C \otimes C \xrightarrow{t \otimes t} C \otimes C \xrightarrow{\mu} C) \\ \eta &= I \xrightarrow{\eta} C\end{aligned}$$

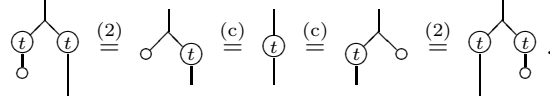
then C is a separable Frobenius monoid in \mathcal{QV} (see Definition A.5).

Proof. We first observe that $(t \otimes t)\delta : C \rightarrow C \otimes C$ and $\epsilon : C \rightarrow I$ are in \mathcal{QV} which follows from (5) and (2) respectively.

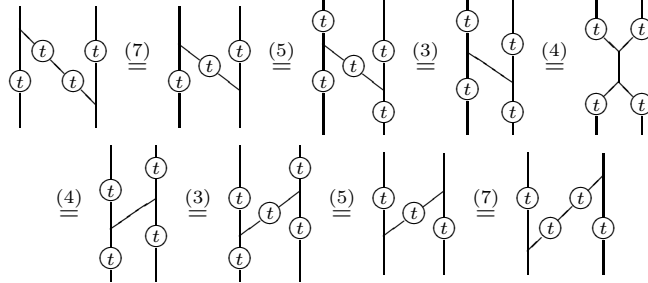
The comonoid identities are given as



and



To see that C is a separable Frobenius monoid we first observe that μ and η are morphisms in \mathcal{QV} from (5) and (2), and the monoid identities are dual to the comonoid identities. The following calculation proves that the Frobenius condition holds.



Finally, that this is a separable Frobenius monoid follows from

$$\mu\delta = \begin{array}{c} \text{Diagram 1} \end{array} \stackrel{(7)}{=} \begin{array}{c} \text{Diagram 2} \end{array} \stackrel{(5)}{=} \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(3)}{=} \begin{array}{c} \text{Diagram 4} \end{array} \stackrel{(6)}{=} \begin{array}{c} \text{Diagram 5} \end{array} = 1_C.$$

□

Corollary 2.4. *Every morphism of weak bimonoids induces an isomorphism on the “objects-of-objects”. That is, if $(A, 1)$ and $(B, 1)$ are weak bimonoids, and $f : (A, 1) \rightarrow (B, 1)$ is a morphism of weak bimonoids, then the induced morphism $tft : (A, t) \rightarrow (B, t)$ is an isomorphism.*

Proof. Note that if $f : A \rightarrow B$ is a morphism of weak bimonoids then $ft = tf$ and $fs = st$. The corollary now follows from Proposition 2.3 and Proposition A.3. □

Proposition 2.5. *If we write C° for the comonoid C with the “opposite” comultiplication defined via*

$$C \xrightarrow{\delta} C \otimes C \xrightarrow{t \otimes t} C \otimes C \xrightarrow{c} C \otimes C = \begin{array}{c} \text{---} \\ | \\ \textcircled{t} \text{---} \textcircled{t} \\ | \quad | \\ \text{---} \end{array}$$

then $s : A \rightarrow C^\circ$ and $t : A \rightarrow C$ are comonoid morphisms. That is, the diagrams

$$\begin{array}{ccc} A & \xrightarrow{s} & C \\ \delta \downarrow & & \downarrow c(t \otimes t)\delta \\ A \otimes A & \xrightarrow{s \otimes s} & C \otimes C \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{t} & C \\ \delta \downarrow & & \downarrow (t \otimes t)\delta \\ A \otimes A & \xrightarrow{t \otimes t} & C \otimes C \end{array}$$

commute.

Proof. The second diagram expresses

$$\begin{array}{c} \textcircled{t} \\ | \\ \textcircled{t} \text{---} \textcircled{t} \\ | \quad | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{t} \text{---} \textcircled{t} \\ | \quad | \\ \text{---} \end{array}$$

which is exactly (5), and the following calculation

$$\begin{array}{c} \text{---} \\ | \\ \textcircled{s} \text{---} \textcircled{s} \\ | \quad | \\ \text{---} \end{array} \stackrel{(5)}{=} \begin{array}{c} \textcircled{s} \\ | \\ \textcircled{s} \text{---} \textcircled{s} \\ | \quad | \\ \text{---} \end{array} \stackrel{(8)}{=} \begin{array}{c} \textcircled{s} \\ | \\ \textcircled{s} \text{---} \textcircled{s} \\ | \quad | \\ \textcircled{t} \text{---} \text{---} \end{array} \stackrel{(3)}{=} \begin{array}{c} \textcircled{s} \\ | \\ \textcircled{t} \text{---} \textcircled{s} \\ | \quad | \\ \text{---} \end{array} \stackrel{(10)}{=} \begin{array}{c} \textcircled{s} \\ | \\ \textcircled{s} \text{---} \textcircled{t} \\ | \quad | \\ \text{---} \end{array} \stackrel{(9)}{=} \begin{array}{c} \textcircled{s} \\ | \\ \textcircled{t} \text{---} \textcircled{t} \\ | \quad | \\ \text{---} \end{array} \stackrel{(8)}{=} \begin{array}{c} \textcircled{s} \\ | \\ \textcircled{t} \text{---} \textcircled{t} \\ | \quad | \\ \text{---} \end{array}$$

shows that the first diagram commutes. \square

3. WEAK HOPF MONOIDS

In this section we introduce weak Hopf monoids. Usually in the literature, a weak Hopf monoid is a weak bimonoid H equipped with an antipode $\nu : H \rightarrow H$ satisfying the three axioms

$$\nu * 1 = t, \quad 1 * \nu = s, \quad \text{and} \quad \nu * 1 * \nu = \nu,$$

where $f * g = \mu(f \otimes g)\delta$ is the convolution product. Our definition is slightly different as, instead of choosing our source morphism in the second axiom, we replace it with

$$1 * \nu = r,$$

where r is defined below. This turns out to be the usual definition of weak Hopf monoids as found in the literature; in the symmetric case see [4], and in the braided case see [1, 2].

3.1. The endomorphism r and weak Hopf monoids. Define an endomorphism $r : A \rightarrow A$ by rotating the target morphism $t : A \rightarrow A$ by π , i.e.,

$$r = \begin{array}{c} \text{---} \\ | \\ \textcircled{t} \\ | \\ \text{---} \end{array}.$$

Since r is just t rotated by π , all the identities for t in Figure 2 rotated by π hold for r . We list some additional identities of r interacting with s and t .

$$(12) \quad \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \textcircled{s} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{s} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \textcircled{s} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array}$$

$$(13) \quad \begin{array}{c} \text{---} \\ | \\ \textcircled{t} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{t} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \textcircled{t} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{t} \\ | \\ \text{---} \end{array}$$

$$(14) \quad \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{s} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{s} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \textcircled{s} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{s} \\ | \\ \text{---} \end{array}$$

The proofs of these properties may also be found in Appendix B.

Definition 3.1. A weak bimonoid H is called a *weak Hopf monoid* if it is equipped with an endomorphism $\nu : H \longrightarrow H$, called the *antipode*, satisfying

$$\begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{t} \\ | \\ \text{---} \end{array}, \quad \begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array}, \quad \begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array}.$$

The axioms of a weak Hopf monoid immediately imply the following identities

$$\begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{t} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array}.$$

The antipode is unique since if ν' is another

$$\nu' = \nu' * 1 * \nu' = t * \nu' = \nu * 1 * \nu' = \nu * r = \nu * 1 * \nu = \nu.$$

If H and K are weak Hopf monoids in \mathcal{V} , then a *morphism of weak Hopf monoids* $f : H \longrightarrow K$ is a morphism $f : H \longrightarrow K$ in \mathcal{V} which is a monoid and comonoid morphism that also preserves the antipode, i.e., $f\nu = \nu f$.

We list some properties of the antipode $\nu : H \longrightarrow H$.

Proposition 3.2.

$$(15) \quad \begin{array}{c} \text{---} \\ | \\ \textcircled{s} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array}$$

$$(16) \quad \begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{t} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{t} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{t} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{t} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{r} \\ | \\ \text{---} \end{array}$$

$$(17) \quad \begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \textcircled{\nu} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array}$$

The last identity (17) states that $\nu : A \longrightarrow A$ is both an anti-comonoid morphism and an anti-monoid morphism.

Proof. The calculation

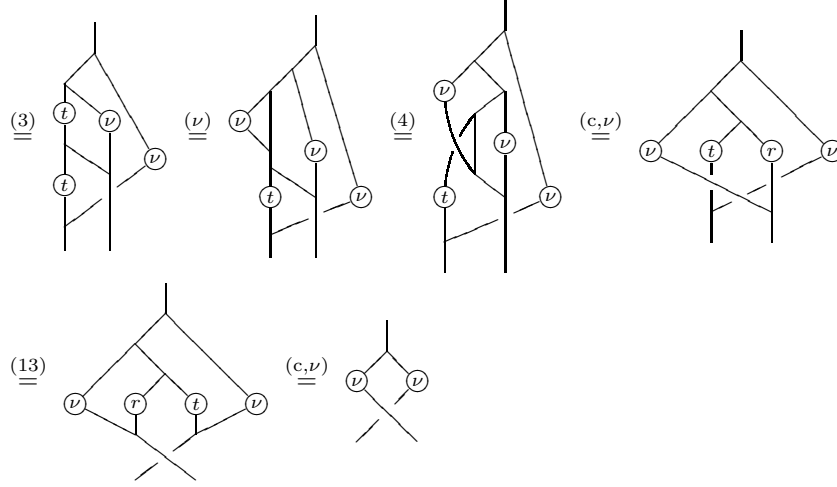
verifies the identity (15), and the following verifies the first identity of (16):

The second identity of (16) follows from a similar calculation.

To prove (17) we will only prove that ν is an anti-comonoid morphism. That ν is an anti-monoid morphism follows by rotating all the diagrams used to prove this statement by π .

The proof of the counit property is easy enough:

The following calculation proves that the antipode is anti-comultiplicative.



□

4. THE MONOIDAL CATEGORY OF A -COMODULES

Suppose $A = (A, 1)$ is a weak bimonoid in $\mathcal{Q}\mathcal{V}$ and let $C = (A, t)$. In this section we describe a monoidal structure on the categories $\mathbf{Bicomod}(C)$ of C -bicomodules in $\mathcal{Q}\mathcal{V}$, and $\mathbf{Comod}(A)$ of right A -comodules in $\mathcal{Q}\mathcal{V}$ such that the underlying functor

$$U : \mathbf{Comod}(A) \longrightarrow \mathbf{Bicomod}(C)$$

is strong monoidal. If A is a weak Hopf monoid then we show that $\mathbf{Comod}_f(A)$, the subcategory consisting of the dualizable objects, is left autonomous.

This section is fairly standard in the $\mathcal{V} = \mathbf{Vect}$ case (see [6], [15], or [16] for example) and carries over rather straightforwardly to the general braided \mathcal{V} case (cf. [10]).

4.1. The monoidal structure on C -bicomodules. Suppose, for this section, that $C \in \mathcal{V}$ is just a comonoid, and that $M \in \mathcal{V}$ is a C -bicomodule with coaction

$$\gamma : M \longrightarrow C \otimes M \otimes C.$$

A left C -coaction and a right C -coaction are obtained from γ by involving the counit ϵ :

$$\begin{aligned} \gamma_l &= (M \xrightarrow{\gamma} C \otimes M \otimes C \xrightarrow{1 \otimes 1 \otimes \epsilon} C \otimes M) \\ \gamma_r &= (M \xrightarrow{\gamma} C \otimes M \otimes C \xrightarrow{\epsilon \otimes 1 \otimes 1} M \otimes C). \end{aligned}$$

Suppose now that N is another C -bicomodule. The tensor product of M and N over C is defined to be the equalizer

$$M \otimes_C N \xrightarrow{\iota} M \otimes N \xrightleftharpoons[1 \otimes \gamma_l]{\gamma_r \otimes 1} M \otimes C \otimes N.$$

Obviously the morphism

$$M \otimes_C N \xrightarrow{\iota} M \otimes N \xrightarrow{\gamma_l \otimes \gamma_r} C \otimes M \otimes N \otimes C$$

$$\gamma : M \otimes_C N \longrightarrow C \otimes M \otimes_C N \otimes C,$$

That this defines a monoidal structure on the category $\mathbf{Bicomod}(C)$ with tensor product \otimes_C and unit C is standard.

Suppose that M is a right A -comodule. We know that $s : A \longrightarrow C^\circ$ and $t : A \longrightarrow C$ are comonoid morphisms and that property (10) holds, where recall that property (10) expresses the commutativity of the following diagram.

$$\begin{array}{ccccc}
 & & A \otimes A & \xrightarrow{s \otimes t} & C \otimes C \\
 & \nearrow \delta & & & \downarrow c \\
 A & & A \otimes A & \xrightarrow{t \otimes s} & C \otimes C \\
 & \searrow \delta & & &
 \end{array}$$

$$\gamma = (M \xrightarrow{\gamma} M \otimes A \xrightarrow{1 \otimes \delta} M \otimes A \otimes A \xrightarrow{c^{-1} \otimes 1} A \otimes M \otimes A \xrightarrow{s \otimes 1 \otimes t} C \otimes M \otimes C),$$

which is

The diagram shows a game tree starting with Nature (N) at the root, who chooses between two states, s and t , from a set C . This leads to an information set for Player M, who chooses between two messages, M and C . This leads to an information set for Player A, who chooses between two actions, A and C . The tree is labeled with $\gamma =$ on the left.

$$\gamma_l = \text{diagram of a vertex } s \text{ with a line to the left and a line to the right that splits into two lines} \quad \text{and} \quad \gamma_r = \text{diagram of a vertex } t \text{ with a line to the left that splits into two lines and a line to the right}$$

Definition 4.1. Let $f, g : X \longrightarrow Y$ be a parallel pair in \mathcal{V} . This pair is called *cosplit* when there is an arrow $d : Y \longrightarrow X$ such that

$$df = 1_X \quad \text{and} \quad fdg = gdg.$$

$$\begin{array}{ccc} X & \xrightarrow{dg} & X \\ & \searrow x \quad \nearrow y & \\ & Q & \end{array} \qquad \begin{array}{ccc} Q & \xrightarrow{1} & Q \\ & \searrow y \quad \nearrow x & \\ & X & \end{array}$$

provides an absolute equalizer (Q, y) for f and g .

Now suppose M and N are A -comodules. Two morphisms $M \otimes N \longrightarrow M \otimes C \otimes N$ are given as

$$\gamma_r \otimes 1 = \begin{array}{c} M \quad N \\ | \quad | \\ \swarrow \quad \downarrow \\ A \\ | \\ \circlearrowleft t \\ | \\ M \quad C \quad N \end{array} \quad \text{and} \quad 1 \otimes \gamma_l = \begin{array}{c} M \quad N \\ | \quad | \\ \downarrow \quad \swarrow \\ \circlearrowleft s \\ | \\ M \quad C \quad N \end{array} \quad A.$$

Proposition 4.2. *The pair $\gamma_r \otimes 1$ and $1 \otimes \gamma_l$ are cosplit by*

$$d = \begin{array}{c} M \quad C \quad N \\ | \quad | \quad | \\ \swarrow \quad \downarrow \\ \circlearrowleft t \\ | \\ \circ \\ | \\ M \quad N \end{array}.$$

Proof. That d is a morphism in \mathcal{QV} follows immediately as t is idempotent. The calculation

$$d(\gamma_r \otimes 1) = \begin{array}{c} | \quad | \\ \swarrow \quad \downarrow \\ \circlearrowleft t \\ | \\ \swarrow \quad \downarrow \\ \circlearrowleft t \\ | \\ \circ \end{array} \stackrel{(7)}{=} \begin{array}{c} | \quad | \\ \swarrow \quad \downarrow \\ \circlearrowleft t \\ | \\ \circ \end{array} \stackrel{(c)}{=} \begin{array}{c} | \quad | \\ \swarrow \quad \downarrow \\ \circlearrowleft t \\ | \\ \circ \end{array} \stackrel{(6)}{=} \begin{array}{c} | \quad | \\ \swarrow \quad \downarrow \\ \circlearrowleft t \\ | \\ \circ \end{array} \stackrel{(c)}{=} \begin{array}{c} | \quad | \\ \swarrow \quad \downarrow \\ \circlearrowleft t \\ | \\ \circ \end{array} = 1_{M \otimes N}$$

shows that $d(\gamma_r \otimes 1) = 1$ and the identity $(\gamma_r \otimes 1)d(1 \otimes \gamma_l) = (1 \otimes \gamma_l)d(1 \otimes \gamma_l)$ follows from:

$$\begin{aligned} (\gamma_r \otimes 1)d(1 \otimes \gamma_l) &= \begin{array}{c} | \quad | \\ \swarrow \quad \downarrow \\ \circlearrowleft s \\ | \\ \swarrow \quad \downarrow \\ \circlearrowleft t \\ | \\ \swarrow \quad \downarrow \\ \circlearrowleft t \\ | \\ \circ \end{array} \stackrel{(8)}{=} \begin{array}{c} | \quad | \\ \swarrow \quad \downarrow \\ \circlearrowleft s \\ | \\ \swarrow \quad \downarrow \\ \circlearrowleft t \\ | \\ \circ \end{array} \stackrel{(2)}{=} \begin{array}{c} | \quad | \\ \swarrow \quad \downarrow \\ \circlearrowleft s \\ | \\ \swarrow \quad \downarrow \\ \circlearrowleft t \\ | \\ \circ \end{array} \stackrel{(c)}{=} \begin{array}{c} | \quad | \\ \swarrow \quad \downarrow \\ \circlearrowleft s \\ | \\ \swarrow \quad \downarrow \\ \circlearrowleft t \\ | \\ \circ \end{array} \stackrel{(12)}{=} \begin{array}{c} | \quad | \\ \swarrow \quad \downarrow \\ \circlearrowleft s \\ | \\ \swarrow \quad \downarrow \\ \circlearrowleft t \\ | \\ \circ \end{array} \\ &\stackrel{(2)}{=} \begin{array}{c} | \quad | \\ \swarrow \quad \downarrow \\ \circlearrowleft s \\ | \\ \swarrow \quad \downarrow \\ \circlearrowleft s \\ | \\ \circ \end{array} \stackrel{(c)}{=} \begin{array}{c} | \quad | \\ \swarrow \quad \downarrow \\ \circlearrowleft s \\ | \\ \swarrow \quad \downarrow \\ \circlearrowleft s \\ | \\ \circ \end{array} \stackrel{(8)}{=} \begin{array}{c} | \quad | \\ \swarrow \quad \downarrow \\ \circlearrowleft s \\ | \\ \swarrow \quad \downarrow \\ \circlearrowleft t \\ | \\ \circ \end{array} = (1 \otimes \gamma_l)d(1 \otimes \gamma_l). \end{aligned}$$

□

The idempotent $d(1 \otimes \gamma_l)$ will be denoted by m . The following calculation gives a simpler representation of m :

$$\text{Diagram 1} \stackrel{(8)}{=} \text{Diagram 2} \stackrel{(2)}{=} \text{Diagram 3} = m.$$

A splitting of m , i.e.,

$$\begin{array}{ccc} (M \otimes N, 1) & \xrightarrow{m} & (M \otimes N, 1) \\ & \searrow m \quad \nearrow m & \\ & (M \otimes N, m) & \end{array} \quad \begin{array}{ccc} (M \otimes N, m) & \xrightarrow{m} & (M \otimes N, m) \\ & \searrow m \quad \nearrow m & \\ & (M \otimes N, 1) & \end{array}$$

provides an absolute equalizer $(M \otimes N, m)$ of $(\gamma_r \otimes 1)$ and $(1 \otimes \gamma_l)$. Thus, the tensor product of M and N over C is

$$M \otimes_C N = (M \otimes N, m).$$

4.3. The coaction on the tensor product. If $\mathbf{Comod}(A)$ is to be a monoidal category with underlying functor $U : \mathbf{Comod}(A) \rightarrow \mathbf{Bicomod}(C)$ strong monoidal, then the tensor product of two A -comodules must also be an A -comodule. In this section we show that the obvious coaction on $M \otimes_C N$, namely,

$$\gamma = \text{Diagram} : M \otimes_C N \rightarrow M \otimes_C N \otimes A$$

does the job.

Lemma 4.3. *The coaction $\gamma : M \otimes_C N \rightarrow M \otimes_C N \otimes A$, as defined above, is a morphism in \mathcal{QV} . That is, the following equation holds.*

$$\text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3}$$

Proof. The first equality is given by

$$\text{Diagram 1} \stackrel{(c)}{=} \text{Diagram 2} \stackrel{(b)}{=} \text{Diagram 3} \stackrel{(c)}{=} \text{Diagram 4},$$

and the second by a similar calculation:

□

Proposition 4.4. $(M \otimes_C N, \gamma)$ is an A -comodule.

Proof. Coassociativity is proved as usual,

and the counit condition as

□

4.4. $\mathbf{Comod}(A)$ is a monoidal category. We now set out to prove the claim at the beginning of this section, that $(\mathbf{Comod}(A), \otimes_C, C)$ is a monoidal category. It will turn out that associativity is a strict equality (if it is so in \mathcal{V}) and the unit conditions are only up to isomorphism.

We state this as a theorem and devote the remainder of this section to its proof.

Theorem 4.5. $\mathbf{Comod}(A) = (\mathbf{Comod}(A), \otimes_C, C)$ is a monoidal category.

First off note that C itself is an A -comodule with coaction

Before proving this theorem it will be useful to have the following lemma.

Lemma 4.6. *The following identities hold.*

Proof. The first identity is proved by

$$\begin{array}{c} \text{Diagram 1} \end{array} \stackrel{(9)}{=} \begin{array}{c} \text{Diagram 2} \end{array} \stackrel{(11)}{=} \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(c)}{=} \begin{array}{c} \text{Diagram 4} \end{array}$$

and the second by

$$\begin{array}{c} \text{Diagram 5} \end{array} \stackrel{(3)}{=} \begin{array}{c} \text{Diagram 6} \end{array} \stackrel{(11)}{=} \begin{array}{c} \text{Diagram 7} \end{array} \stackrel{(c)}{=} \begin{array}{c} \text{Diagram 8} \end{array}.$$

□

Proof of Theorem 4.5. Consider $(M \otimes_C N) \otimes_C P$ and $M \otimes_C (N \otimes_C P)$ in \mathcal{QV} . The former is $(M \otimes N \otimes P, u)$ and the latter $(M \otimes N \otimes P, v)$ where

$$u = \begin{array}{c} \text{Diagram 9} \end{array} \quad \text{and} \quad v = \begin{array}{c} \text{Diagram 10} \end{array}.$$

Since, by Lemma 4.3, γ is a morphism in \mathcal{QV} , both u and v may be rewritten as

$$\begin{array}{c} \text{Diagram 11} \end{array}$$

proving the (strict) equality $(M \otimes_C N) \otimes_C P = M \otimes_C (N \otimes_C P)$ in \mathcal{QV} (since we are writing as if \mathcal{V} were strict).

It remains to prove $M \otimes_C C \cong M \cong C \otimes_C M$. By definition

$$M \otimes_C C = (M \otimes C, \begin{array}{c} \text{Diagram 12} \end{array}) \quad \text{and} \quad C \otimes_C M = (C \otimes M, \begin{array}{c} \text{Diagram 13} \end{array}).$$

We will show that the morphisms

$$\begin{array}{c} \text{Diagram 14} \end{array} : M \otimes_C C \longrightarrow M \quad \text{and} \quad \begin{array}{c} \text{Diagram 15} \end{array} : M \longrightarrow M \otimes_C C$$

will establish the isomorphism $M \otimes_C C \cong M$, and

$$\begin{array}{c} \text{Diagram 16} \end{array} : C \otimes_C M \longrightarrow M \quad \text{and} \quad \begin{array}{c} \text{Diagram 17} \end{array} : M \longrightarrow C \otimes_C M$$

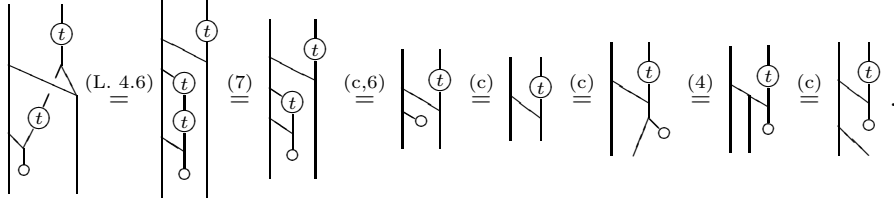
the isomorphism $M \cong C \otimes_C M$. These morphisms are easily seen to be in \mathcal{QV} , and the fact that they are mutually inverse pairs is given in one direction by Lemma 4.6, and in the other by an easy string calculation making use of the identity (6).

It now remains to show that these four morphisms are A -comodule morphisms, i.e., that they are in $\mathbf{Comod}(A)$. Note that $M \otimes_C C$ and $C \otimes_C M$ are A -comodules via the coactions

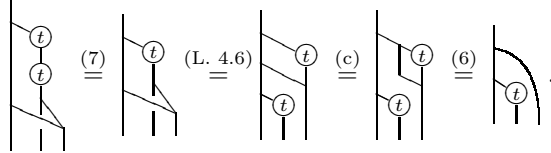


respectively. We then have:

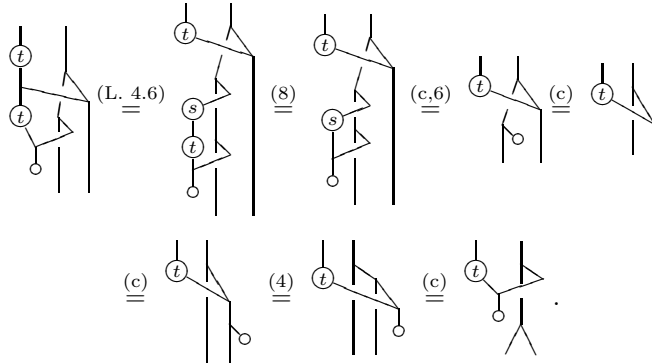
- $\begin{array}{c} | \\ | \\ \circ \end{array} \begin{array}{c} t \\ | \\ \circ \end{array} : M \otimes_C C \longrightarrow M$ is an A -comodule morphism as:

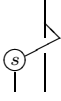


- $\begin{array}{c} | \\ | \\ \circ \end{array} \begin{array}{c} t \\ | \\ \circ \end{array} : M \longrightarrow M \otimes_C C$ is an A -comodule morphism as:



- $\begin{array}{c} | \\ | \\ \circ \end{array} \begin{array}{c} t \\ | \\ \circ \end{array} : C \otimes_C M \longrightarrow M$ is an A -comodule morphism as:



•  : $M \longrightarrow C \otimes_C M$ is an A -comodule morphism as:

$$\begin{array}{c} \text{Diagram 1} \end{array} \stackrel{(L, 4.6)}{=} \begin{array}{c} \text{Diagram 2} \end{array} \stackrel{(8)}{=} \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(c, 6)}{=} \begin{array}{c} \text{Diagram 4} \end{array}.$$

The diagrams are as follows: Diagram 1 is a vertical line with a circle 's' in the middle. Diagram 2 is a vertical line with a circle 's' in the middle and a circle 't' in the middle. Diagram 3 is a vertical line with a circle 's' in the middle and a circle 't' in the middle, with a diagonal line connecting the top of the 's' circle to the top of the 't' circle. Diagram 4 is a vertical line with a circle 's' in the middle.

Thus, $M \otimes_C C \cong M \cong C \otimes_C M$ in \mathcal{QV} . □

Thus, $\mathbf{Comod}(A) = (\mathbf{Comod}(A), \otimes_C, C)$ is a monoidal category.

4.5. The forgetful functor from A -comodules to C -bicomodules. There is a forgetful functor $U : \mathbf{Comod}(A) \longrightarrow \mathbf{Bicomod}(C)$ which assigns to each A -comodule M a C -bicomodule UM which is M itself with coaction

$$\begin{array}{c} M \\ \swarrow \quad \searrow \\ \text{Diagram} \\ \swarrow \quad \searrow \\ C \quad M \quad C \end{array}.$$

The diagram shows a vertical line with a circle 's' in the middle and a circle 't' in the middle. A diagonal line connects the top of the 's' circle to the top of the 't' circle. The labels 'C', 'M', and 'C' are at the bottom, and 'M' and 'A' are at the top.

A morphism of A -comodules $f : M \longrightarrow N$ is automatically a morphism of the underlying C -bicomodules $f : UM \longrightarrow UN$.

Proposition 4.7. *The forgetful functor $U : \mathbf{Comod}(A) \longrightarrow \mathbf{Bicomod}(C)$ is strong monoidal.*

Proof. We must establish the C -bicomodule isomorphisms

$$C \cong UC \quad \text{and} \quad UM \otimes_C UN \cong U(M \otimes_C N).$$

The first is obvious. To establish the second isomorphism we observe that the object $UM \otimes_C UN$ is $(M \otimes_C N, m)$ with coaction

$$\begin{array}{c} \text{Diagram} \\ \swarrow \quad \searrow \\ \text{Diagram} \\ \swarrow \quad \searrow \\ C \quad M \quad C \end{array}.$$

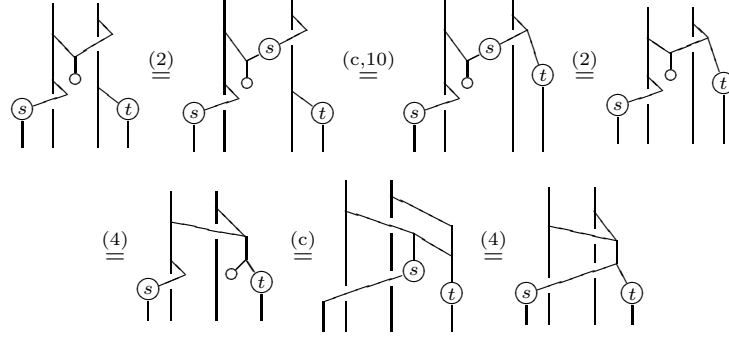
The diagram shows a vertical line with a circle 's' in the middle and a circle 't' in the middle. A diagonal line connects the top of the 's' circle to the top of the 't' circle. The labels 'C', 'M', and 'C' are at the bottom, and 'M' and 'A' are at the top.

and $U(M \otimes_C N)$ is also $(M \otimes_C N, m)$ but with coaction

$$\begin{array}{c} \text{Diagram} \\ \swarrow \quad \searrow \\ \text{Diagram} \\ \swarrow \quad \searrow \\ C \quad M \quad C \end{array}.$$

The diagram shows a vertical line with a circle 's' in the middle and a circle 't' in the middle. A diagonal line connects the top of the 's' circle to the top of the 't' circle. The labels 'C', 'M', and 'C' are at the bottom, and 'M' and 'A' are at the top.

The following calculation shows that these two coactions are the same, and hence the isomorphism $U(M \otimes_C N) \cong UM \otimes_C UN$.



□

This may seem to be a strict equality, but as tensor products are really only defined up to isomorphism we prefer “strong”.

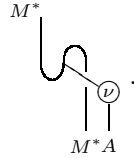
4.6. $\mathbf{Comod}_f(H)$ is left autonomous. Let \mathcal{V}_f denote the subcategory of \mathcal{V} consisting of the objects with a left dual (since \mathcal{V} is braided left duals are right duals). There is a forgetful functor $U_l : \mathbf{Comod}(H) \rightarrow \mathcal{V}$ defined as the composite of the two forgetful functors $\mathbf{Comod}(H) \rightarrow \mathbf{Bicomod}(C)$ and $\mathbf{Bicomod}(C) \rightarrow \mathcal{V}$. This composite $U_l : \mathbf{Comod}(H) \rightarrow \mathcal{V}$ is sometimes called the *long forgetful functor*, as opposed to the *short forgetful functor* $U : \mathbf{Comod}(H) \rightarrow \mathbf{Bicomod}(C)$.

Let us say an object $M \in \mathbf{Comod}(H)$ is *dualizable* if $U_l M$ has a left dual in \mathcal{V} , i.e., $U_l M \in \mathcal{V}_f$. Denote by $\mathbf{Comod}_f(H)$ the subcategory of $\mathbf{Comod}(H)$ consisting of the dualizable objects.

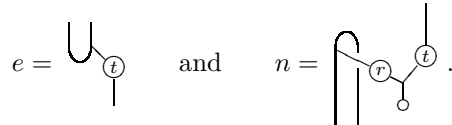
The goal of this section is to prove the following proposition.

Proposition 4.8. *If H is a weak Hopf monoid then the category $\mathbf{Comod}_f(H)$ is left autonomous (= left rigid = left compact).*

Suppose $M \in \mathbf{Comod}_f(H)$ has a left dual M^* in \mathcal{V} . Using the antipode of H a coaction on M^* is defined as



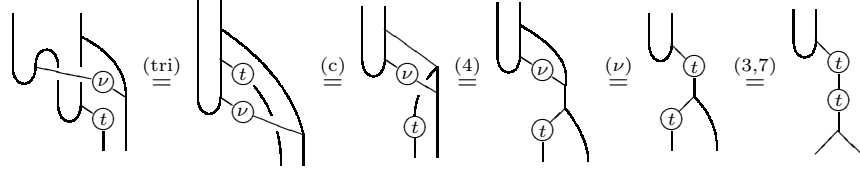
By (17) it is easy to see that this defines a comodule structure on M^* . We claim that M^* is the left dual of M in $\mathbf{Comod}_f(H)$. Define morphisms $e : M^* \otimes_C M \rightarrow C$ and $n : C \rightarrow M \otimes_C M^*$ via



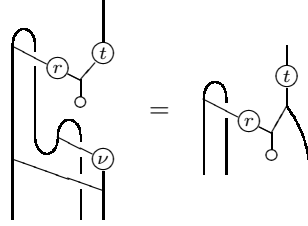
Proposition 4.9. *Suppose $M \in \mathbf{Comod}_f(H)$ with underlying left dual M^* . Then M^* with evaluation and coevaluation morphisms e and n respectively is the left dual of M in $\mathbf{Comod}_f(H)$. That is, $\mathbf{Comod}_f(H)$ is left autonomous.*

Proof. Let M , M^* , e , and n be as above. We will first show that e and n are comodule morphisms, and secondly that they satisfy the triangle identities.

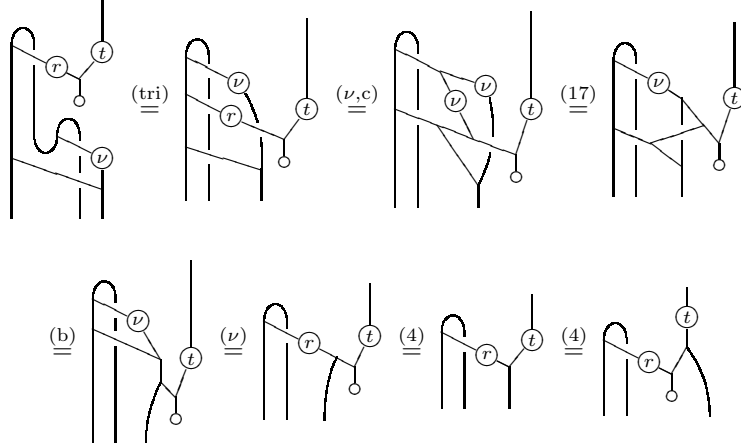
The following calculation shows that e is a comodule morphism.



To show that n is a comodule morphism we must establish the equality



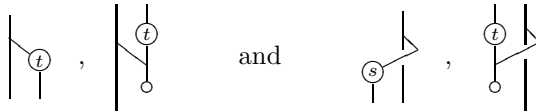
which is proved by the following calculation.



It remains to show that e and n satisfy the triangle identities, i.e., that the following composites are the identity:

$$\begin{aligned} \text{(i)} \quad M &\cong C \otimes_C M \xrightarrow{n \otimes 1} M \otimes_C M^* \otimes_C M \xrightarrow{1 \otimes e} M \otimes_C C \cong M \\ \text{(ii)} \quad M^* &\cong M^* \otimes_C C \xrightarrow{1 \otimes n} M^* \otimes_C M \otimes_C M^* \xrightarrow{e \otimes 1} C \otimes_C M^* \cong M^* . \end{aligned}$$

Recall that $M \cong M \otimes_C C$ and $M \cong C \otimes_C M$ via



respectively.

The following calculation proves (i):

$$\begin{array}{c}
 \text{Diagram 1} \xrightarrow{(\text{tri})} \text{Diagram 2} \xrightarrow{(\text{c},13)} \text{Diagram 3} \xrightarrow{(2,c)} \text{Diagram 4} \xrightarrow{(6)} = 1_M,
 \end{array}$$

and (ii) is given by:

$$\begin{array}{c}
 \text{Diagram 1} \xrightarrow{(\text{tri})} \text{Diagram 2} \xrightarrow{(13)} \text{Diagram 3} \xrightarrow{(2,\nu)} \text{Diagram 4} \xrightarrow{(2,c)} \text{Diagram 5} \\
 \xrightarrow{(13)} \text{Diagram 6} \xrightarrow{(2)} \text{Diagram 7} \xrightarrow{(\nu)} \text{Diagram 8} \xrightarrow{(2,c)} \text{Diagram 9} \xrightarrow{(\text{tri})} = 1_{M^*}.
 \end{array}$$

This completes the proof that M^* is the left dual of M in $\mathbf{Comod}_f(H)$, and hence that $\mathbf{Comod}_f(H)$ is left autonomous. \square

5. FROBENIUS MONOID EXAMPLE

Let R be a separable Frobenius monoid in \mathcal{V} . In this section we prove that $R \otimes R$ is an example of a weak Hopf monoid with an invertible antipode. In the case $\mathcal{V} = \mathbf{Vect}$, this example is essentially the same as in [4, Appendix].

Let R be a Frobenius monoid in \mathcal{V} . Then $R \otimes R$ becomes a comonoid via

$$\delta = \begin{array}{c} \diagup \\ \diagdown \end{array} \quad \text{and} \quad \epsilon = \cup.$$

(where, for simplicity, in this section we will adopt the simpler notation

$$\begin{array}{c} \circ \\ \diagup \quad \diagdown \end{array} = \cap \quad \text{and} \quad \begin{array}{c} \diagup \quad \diagdown \\ \circ \end{array} = \cup),$$

and a monoid via

$$\mu = \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \quad \text{and} \quad \eta = \begin{array}{c} \circ \quad \circ \\ \diagdown \quad \diagup \end{array}.$$

The comonoid structure is via the comonad generated by the adjunction $R \dashv R$. The monoid structure is the usual monoid structure (viewing R as a monoid) on the tensor product $R^\circ \otimes R$, where R° is the opposite monoid of R .

Proposition 5.1. *If R is separable, meaning $\mu\delta = 1_R$, then $R \otimes R$ is a weak bimonoid. An invertible antipode ν on $R \otimes R$ is given by*

$$\nu = \text{diagram of a crossing with a loop on the left strand}.$$

which makes $R \otimes R$ into a weak Hopf monoid.

The following three sets of calculations establish respectively the axioms (b), (v), and (w), and hence the first claim.

The axiom (b) is given by:

$$(\mu \otimes \mu)(1 \otimes c \otimes 1)(\delta \otimes \delta) = \text{diagram} \stackrel{(\text{nat})}{=} \text{diagram} \stackrel{(\text{sep})}{=} \text{diagram} = \delta\mu.$$

Axiom (v) is seen from the diagrams:

$$\begin{aligned} & \text{diagram of a comultiplication} : \text{diagram of a multiplication with a loop} \\ & \text{diagram of a comultiplication} : \text{diagram} \stackrel{(\text{c}, \text{tri})}{=} \text{diagram} \stackrel{(\text{c})}{=} \text{diagram} \\ & \text{diagram of a comultiplication} : \text{diagram} \stackrel{(\text{c})}{=} \text{diagram} \stackrel{(\text{tri})}{=} \text{diagram} \\ & \text{diagram of a comultiplication} \stackrel{(\text{c})}{=} \text{diagram} \end{aligned}$$

For (w), by the naturality of the braiding and the counit property of R each equation in (w), i.e.,

$$\text{diagram of a comultiplication} , \quad \text{diagram of a multiplication} , \quad \text{diagram of a comultiplication with a loop}$$

is easily seen to be equal to the following diagram

$$\begin{array}{c} \circ \\ | \\ \cap \\ | \\ \circ \end{array}.$$

Thus, $R \otimes R$ is a weak bimonoid. We next prove that that $R \otimes R$ is a weak Hopf monoid with invertible antipode

$$\nu = \begin{array}{c} \diagup \diagdown \\ | \\ \cap \\ | \\ \circ \end{array}.$$

An inverse to ν is easily seen to be given by

$$\nu^{-1} = \begin{array}{c} \diagdown \diagup \\ | \\ \cap \\ | \\ \circ \end{array},$$

and so the antipode is invertible. We note that (in simplified form)

$$r = \begin{array}{c} \diagup \diagdown \\ | \\ \cap \\ | \\ \circ \end{array} \quad \text{and} \quad t = \begin{array}{c} \circ \\ | \\ \cap \\ | \\ \diagup \diagdown \end{array}.$$

The following calculations then prove the antipode axioms.

$$\mu(\nu \otimes 1)\delta = \begin{array}{c} \diagup \diagdown \\ | \\ \cap \\ | \\ \circ \end{array} \begin{array}{c} \diagup \diagdown \\ | \\ \cap \\ | \\ \diagup \diagdown \end{array} \stackrel{(\text{tri})}{=} \begin{array}{c} \circ \\ | \\ \cap \\ | \\ \diagup \diagdown \end{array} \stackrel{(\text{sep})}{=} \begin{array}{c} \circ \\ | \\ \cap \\ | \\ \diagup \diagdown \end{array} = t$$

$$\mu(1 \otimes \nu)\delta = \begin{array}{c} \diagup \diagdown \\ | \\ \cap \\ | \\ \circ \end{array} \begin{array}{c} \diagup \diagdown \\ | \\ \cap \\ | \\ \diagup \diagdown \end{array} \stackrel{(\text{nat})}{=} \begin{array}{c} \diagup \diagdown \\ | \\ \cap \\ | \\ \circ \end{array} \stackrel{(\text{sep})}{=} \begin{array}{c} \diagup \diagdown \\ | \\ \cap \\ | \\ \circ \end{array} = r$$

$$\mu_3(\nu \otimes 1 \otimes \nu)\delta_3 = \begin{array}{c} \diagup \diagdown \\ | \\ \cap \\ | \\ \circ \end{array} \begin{array}{c} \diagup \diagdown \\ | \\ \cap \\ | \\ \diagup \diagdown \end{array} \begin{array}{c} \diagup \diagdown \\ | \\ \cap \\ | \\ \diagup \diagdown \end{array} \stackrel{(\text{sep})}{=} \begin{array}{c} \diagup \diagdown \\ | \\ \cap \\ | \\ \circ \end{array} \stackrel{(\text{sep})}{=} \begin{array}{c} \diagup \diagdown \\ | \\ \cap \\ | \\ \circ \end{array} \stackrel{(\text{c})}{=} \begin{array}{c} \diagup \diagdown \\ | \\ \cap \\ | \\ \circ \end{array} = \nu$$

Thus, $R \otimes R$ is a weak Hopf monoid with invertible antipode.

6. QUANTUM GROUPOIDS

In this section we recall the quantum categories and quantum groupoids of Day and Street [9]. There is a succinct definition given in [9, p. 216] in terms of “basic data” and “Hopf basic data”. Here we give the unpacked definition of quantum category and quantum groupoid which is essentially found in [9, p. 221]; however, we do make a correction.

Our setting is a braided monoidal category $\mathcal{V} = (\mathcal{V}, \otimes, I, c)$ in which the functors

$$A \otimes - : \mathcal{V} \longrightarrow \mathcal{V}$$

with $A \in \mathcal{V}$, preserve coreflexive equalizers, i.e., equalizers of pairs of morphisms with a common left inverse.

6.1. Quantum categories. Suppose A and C are comonoids in \mathcal{V} and $s : A \longrightarrow C^\circ$ and $t : A \longrightarrow C$ are comonoid morphisms such that the diagram

$$\begin{array}{ccccc} & & A \otimes A & \xrightarrow{s \otimes t} & C \otimes C \\ & \nearrow \delta & & & \downarrow c \\ A & & & & \\ & \searrow \delta & A \otimes A & \xrightarrow{t \otimes s} & C \otimes C \end{array}$$

commutes. Then A may be viewed as a C -bicomodule with left and right coactions defined respectively via

$$\begin{aligned} \gamma_l &= (A \xrightarrow{\delta} A \otimes A \xrightarrow{1 \otimes s} A \otimes C \xrightarrow{c^{-1}} C \otimes A) \\ \gamma_r &= (A \xrightarrow{\delta} A \otimes A \xrightarrow{1 \otimes t} A \otimes C). \end{aligned}$$

Recall that the tensor product $P = A \otimes_C A$ of A with itself over C is defined as the equalizer

$$P \xrightarrow{\iota} A \otimes A \xrightleftharpoons[1 \otimes \gamma_l]{\gamma_r \otimes 1} A \otimes C \otimes A.$$

The following diagrams may be seen to commute

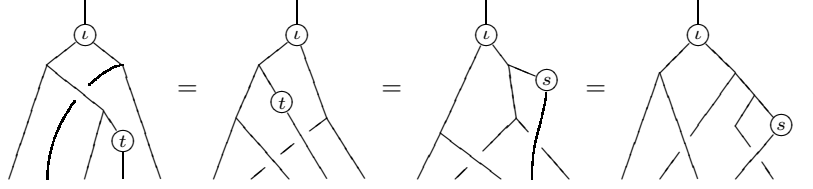
$$\begin{aligned} P &\xrightarrow{\iota} A \otimes A \xrightarrow{\gamma_l \otimes 1} C \otimes A \otimes A \xrightleftharpoons[1 \otimes 1 \otimes \gamma_l]{1 \otimes \gamma_r \otimes 1} C \otimes A \otimes C \otimes A \\ P &\xrightarrow{\iota} A \otimes A \xrightarrow{1 \otimes \gamma_r} A \otimes A \otimes C \xrightleftharpoons[1 \otimes \gamma_l \otimes 1]{\gamma_r \otimes 1 \otimes 1} A \otimes C \otimes A \otimes C \end{aligned}$$

and therefore induce respectively a left C - and right C -coaction on P . These coactions make P into a C -bicomodule.

The commutativity of the diagram

$$P \xrightarrow{\iota} A \otimes A \xrightarrow{\delta \otimes \delta} A^{\otimes 4} \xrightarrow{1 \otimes c \otimes 1} A^{\otimes 4} \xrightleftharpoons[1 \otimes 1 \otimes 1 \otimes \gamma_l]{1 \otimes 1 \otimes \gamma_r \otimes 1} A \otimes A \otimes A \otimes C \otimes A$$

may be seen from



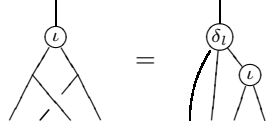
and as $1 \otimes 1 \otimes \iota$ is the equalizer of $1 \otimes 1 \otimes \gamma_r \otimes 1$ and $1 \otimes 1 \otimes 1 \otimes \gamma_l$, there is a unique morphism

$$\delta_l : P \longrightarrow A \otimes A \otimes P$$

making the diagram

$$\begin{array}{ccccc} P & \xrightarrow{\iota} & A \otimes A & \xrightarrow{\delta \otimes \delta} & A \otimes A \otimes A \otimes A \\ \delta_l \downarrow & & & & \downarrow 1 \otimes c \otimes 1 \\ A \otimes A \otimes P & \xrightarrow{1 \otimes 1 \otimes \iota} & A \otimes A \otimes A \otimes A & & \end{array}$$

commute. In strings,



It is easy to see (postcompose with the monomorphism $1 \otimes 1 \otimes 1 \otimes 1 \otimes \iota$) that the morphism δ_l is the left coaction of the comonoid $A \otimes A$ on P making P into a (left) $A \otimes A$ -comodule. This means that the diagrams

$$\begin{array}{ccc} P & \xrightarrow{\delta_l} & A \otimes A \otimes P \\ \delta_l \downarrow & & \downarrow 1 \otimes 1 \otimes \delta_l \\ A \otimes A \otimes P & & A \otimes A \otimes A \otimes P \\ \delta \otimes \delta \otimes 1 \downarrow & & \downarrow 1 \otimes c \otimes 1 \otimes 1 \\ A \otimes A \otimes A \otimes A \otimes P & \xrightarrow{1 \otimes c \otimes 1 \otimes 1} & A \otimes A \otimes A \otimes A \otimes P \end{array} \quad \begin{array}{ccc} P & \xrightarrow{\delta_l} & A \otimes A \otimes P \\ & \searrow 1 & \downarrow \epsilon \otimes \epsilon \otimes 1 \\ & & P \end{array}$$

commute.

We are now ready to state the definition. A *quantum category* in \mathcal{V} consists of the data $\mathbf{A} = (A, C, s, t, \mu, \eta)$ where A, C, s, t are as above, and $\mu : P = A \otimes_C A \longrightarrow A$ and $\eta : C \longrightarrow A$ are morphisms in \mathcal{V} , called the *composition morphism* and *unit morphism* respectively. This data must satisfy axioms (B1) through (B6) below.

(B1) (A, μ, η) is a monoid in $\mathbf{Bicomod}(C)$.

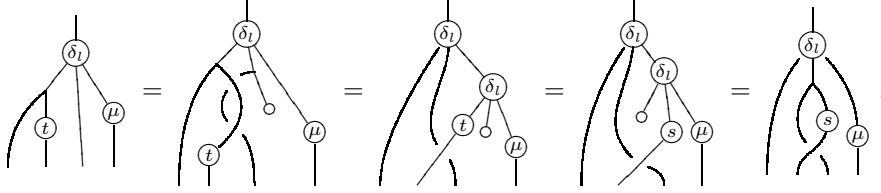
(B2) The following diagram commutes.

$$P \xrightarrow{\delta_l} A \otimes A \otimes P \xrightarrow[\epsilon \otimes s \otimes 1]{t \otimes \epsilon \otimes 1} C \otimes P \xrightarrow{1 \otimes \mu} C \otimes A$$

Before stating (B3), we use (B2) to show that the diagram

$$P \xrightarrow{\delta_l} A \otimes A \otimes P \xrightarrow{1 \otimes 1 \otimes \mu} A \otimes A \otimes A \xrightleftharpoons[1 \otimes \gamma_l \otimes 1]{\gamma_r \otimes 1 \otimes 1} A \otimes C \otimes A \otimes A$$

commutes, as seen by the calculation



As $\iota \otimes 1$ is the equalizer of $\gamma_r \otimes 1 \otimes 1$ and $1 \otimes \gamma_l \otimes 1$ there is a unique morphism $\delta_r : P \rightarrow P \otimes A$ making the square

$$\begin{array}{ccc} P & \xrightarrow{\delta_l} & A \otimes A \otimes P \\ \delta_r \downarrow & & \downarrow 1 \otimes 1 \otimes \mu \\ P \otimes A & \xrightarrow{\iota \otimes 1} & A \otimes A \otimes A \end{array}$$

commute. We can now state (B3).

(B3) The following diagram commutes.

$$\begin{array}{ccc} P & \xrightarrow{\mu} & A \\ \delta_r \downarrow & & \downarrow \delta \\ P \otimes A & \xrightarrow{\mu \otimes 1} & A \otimes A \end{array}$$

(B4) The following diagram commutes.

$$\begin{array}{ccc} P & \xrightarrow{\mu} & A \\ \iota \downarrow & & \downarrow \epsilon \\ A \otimes A & \xrightarrow{\epsilon \otimes \epsilon} & I \end{array}$$

(B5) The following diagram commutes.

$$\begin{array}{ccc} C & & \\ \eta \downarrow & \searrow \epsilon & \\ A & & I \end{array}$$

(B6) The following diagram commutes.

$$\begin{array}{ccccc}
 & A & \xrightarrow{\delta} & A \otimes A & \xrightarrow{s \otimes 1} & C \otimes A \\
 \eta \nearrow & & & & & \searrow \eta \otimes 1 \\
 C & \xrightarrow{\eta} & A & \xrightarrow{\delta} & A \otimes A & \\
 \eta \searrow & & & & & \nearrow \eta \otimes 1 \\
 & A & \xrightarrow{\delta} & A \otimes A & \xrightarrow{t \otimes 1} & C \otimes A
 \end{array}$$

A consequence of these axioms is that P becomes a left $A \otimes A$ -, right A -bicomodule.

The axiom (B6) makes C into a right A -comodule via

$$C \xrightarrow{\eta} A \xrightarrow{\delta} A \otimes A \xrightarrow{s \otimes 1} C \otimes A.$$

We refer to A as the *object-of-arrows* and C as the *object-of-objects*.

6.2. Quantum groupoids. Suppose we have comonoid isomorphisms

$$v : C^{\circ\circ} \xrightarrow{\cong} C \quad \text{and} \quad \nu : A^\circ \xrightarrow{\cong} A.$$

Denote by P_l the left $A^{\otimes 3}$ -comodule P with coaction defined by

$$P \xrightarrow{\delta} A \otimes A \otimes P \otimes A \xrightarrow{1 \otimes 1 \otimes 1 \otimes \nu} A \otimes A \otimes P \otimes A \xrightarrow{1 \otimes 1 \otimes c_{P,A}} A \otimes A \otimes A \otimes P,$$

and by P_r the left $A^{\otimes 3}$ -comodule P with coaction defined by

$$P \xrightarrow{\delta} A \otimes A \otimes P \otimes A \xrightarrow{1 \otimes 1 \otimes 1 \otimes \nu^{-1}} A \otimes A \otimes P \otimes A \xrightarrow{c_{A \otimes A \otimes P, A}^{-1}} A \otimes A \otimes A \otimes P.$$

Furthermore, suppose that $\theta : P_l \rightarrow P_r$ is a left $A^{\otimes 3}$ -comodule isomorphism. We define a *quantum groupoid* in \mathcal{V} to be a quantum category \mathbf{A} in \mathcal{V} equipped with an v , ν , and θ satisfying (G1) through (G3) below.

(G1) $s\nu = t$,

(G2) $t\nu = vs$, and

(G3) the diagram³

$$\begin{array}{ccc}
 P & \xrightarrow{\varsigma} & C \otimes C \otimes C \xrightarrow{c_{C,C \otimes C}} C \otimes C \otimes C \\
 \theta \downarrow & & \downarrow 1 \otimes 1 \otimes \nu \\
 P & \xrightarrow{\varsigma} & C \otimes C \otimes C
 \end{array}$$

commutes, where the morphism $\varsigma : P \rightarrow C^{\otimes 3}$ is defined by taking either of the equal routes

$$P \xrightarrow{\iota} A \otimes A \xrightarrow[1 \otimes \gamma_l]{\gamma_r \otimes 1} A \otimes C \otimes A \xrightarrow{s \otimes 1 \otimes t} C^{\otimes 3}.$$

³This corrects [9, §12, p. 223].

7. WEAK HOPF MONOIDS ARE QUANTUM GROUPOIDS

The goal of this section is to prove the following theorem.

Theorem 7.1. *A weak bimonoid in \mathcal{QV} is a quantum category in \mathcal{QV} whose object-of-objects is a separable Frobenius monoid. If the weak bimonoid is equipped with an invertible antipode, making it a weak Hopf monoid, then the quantum category becomes a quantum groupoid.*

7.1. Weak bimonoids are quantum categories. Let $A = (A, 1)$ be a weak bimonoid in \mathcal{QV} with source morphism s and target morphism t and set $C = (A, t)$. This data along with

$$\mu = \begin{array}{c} \diagup \quad \diagdown \\ | \\ \text{---} \end{array} : P \longrightarrow A$$

$$\eta = t : C \longrightarrow A$$

forms a quantum category in \mathcal{QV} . The morphisms s and t are obviously in \mathcal{QV} , hence so is $\eta = t$, and

$$\begin{array}{c} \diagup \quad \diagdown \\ | \\ \text{---} \end{array} \stackrel{(\text{nat})}{=} \begin{array}{c} \diagup \quad \diagdown \\ | \\ \text{---} \end{array} \stackrel{(\text{b})}{=} \begin{array}{c} \diagup \quad \diagdown \\ | \\ \text{---} \end{array}$$

shows that μ is as well. Recall that $P = (A \otimes A, m)$ where

$$m = \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \text{---} \end{array}.$$

The morphisms $\delta_l : P \longrightarrow A \otimes A \otimes P$ and $\delta_r : P \longrightarrow P \otimes A$ are given by

$$\delta_l = \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \text{---} \end{array} \quad \delta_r = \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \text{---} \end{array}.$$

The two calculations

$$\begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \text{---} \end{array} \stackrel{(\text{c})}{=} \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \text{---} \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \text{---} \end{array}$$

and

$$\begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \text{---} \end{array} \stackrel{(\text{c})}{=} \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \text{---} \end{array} \stackrel{(\text{b})}{=} \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \text{---} \end{array}$$

show that these are morphisms in \mathcal{QV} .

To see that (A, μ, η) is a comonoid in $\mathbf{Bicomod}(C)$ notice that associativity follows from the associativity of the μ viewed as a weak bimonoid and the counit property may be seen from property (6), i.e.,

$$\mu(1 \otimes t)\delta = 1_A \quad \text{and} \quad \mu(s \otimes 1)c^{-1}\delta = 1_A,$$

and so (B1) holds. (B2) follows from one application of (12), (B3) from (b), (B4) from (c), and (B5) from (2), while the calculation

verifies (B6). Thus, $\mathbf{A} = (A, C, s, t, \mu, \eta)$ is a quantum category in \mathcal{QV} .

7.2. Weak Hopf monoids are quantum groupoids. Now suppose that $A = (A, 1)$ is a weak Hopf monoid in \mathcal{QV} with an invertible antipode $\nu : A \longrightarrow A$, and that $\mathbf{A} = (A, C, s, t, \mu, \eta)$ is as above. The data for a quantum groupoid (v, ν, θ) is

$$\begin{aligned} v &= tvvt : C^{\circ\circ} \longrightarrow C \\ \nu &= \nu : A^{\circ} \longrightarrow A \\ \theta &= \text{[string diagram]} : P \longrightarrow P. \end{aligned}$$

In the remainder of this section we will verify this claim.

The morphisms v and ν are obviously morphisms in \mathcal{QV} , and the two calculations

and

$$\stackrel{(16)}{=} \text{diagram} \stackrel{(2)}{=} \text{diagram} \stackrel{(c)}{=} \text{diagram} \stackrel{(\nu)}{=} \text{diagram} \stackrel{(2,c)}{=} \text{diagram}$$

show that θ is as well.

Lemma 7.2. *An inverse for θ is given by*

$$\theta^{-1} = \text{diagram},$$

Proof. Since

$$\text{diagram} \stackrel{(c)}{=} \text{diagram} \stackrel{(b)}{=} \text{diagram}$$

it is clear that θ^{-1} is a morphism in \mathcal{QV} .

That θ^{-1} is an inverse for θ may be seen in one direction from

$$\begin{aligned} \theta^{-1}\theta &= \text{diagram} \stackrel{(17)}{=} \text{diagram} = \text{diagram} \stackrel{(c)}{=} \text{diagram} \\ &\stackrel{(\dagger)}{=} \text{diagram} \stackrel{(4)}{=} \text{diagram} \stackrel{(2)}{=} \text{diagram} = 1_P \end{aligned}$$

where (\dagger) is given by

$$\text{diagram} = \text{diagram} \stackrel{(17)}{=} \text{diagram} \stackrel{(\nu)}{=} \text{diagram} \stackrel{(15)}{=} \text{diagram},$$

and in the other direction by:

$$\begin{aligned}
 \theta\theta^{-1} &= \text{[Diagram 1]} \stackrel{(\ddagger)}{=} \text{[Diagram 2]} \stackrel{(c)}{=} \text{[Diagram 3]} \stackrel{(17)}{=} \text{[Diagram 4]} \\
 &= \text{[Diagram 5]} \stackrel{(c,\nu)}{=} \text{[Diagram 6]} \stackrel{(4)}{=} \text{[Diagram 7]} \stackrel{(2)}{=} \text{[Diagram 8]} = 1_P
 \end{aligned}$$

for which the first step (\ddagger) holds since θ is a morphism in \mathcal{QV} . \square

That the antipode $\nu : A^\circ \longrightarrow A$ is a comonoid isomorphism is our assumption. That $v : C^{\circ\circ} \longrightarrow C$ is as well may be seen from the following calculation:

$$\begin{aligned}
 (t \otimes t)\delta v &= (t \otimes t)\delta t\nu t \\
 &= (t \otimes t)\delta\nu t & (3) \\
 &= (t \otimes t)c(\nu \otimes \nu)\delta\nu t & (17) \\
 &= (t \otimes t)c(\nu \otimes \nu)c(\nu \otimes \nu)\delta t & (17) \\
 &= (t \otimes t)(\nu \otimes \nu)(\nu \otimes \nu)cc\delta t & (\text{nat}) \\
 &= (t \otimes t)(t \otimes t)(\nu \otimes \nu)(\nu \otimes \nu)cc\delta t & (7) \\
 &= (t \otimes t)(\nu \otimes \nu)(r \otimes r)(\nu \otimes \nu)cc\delta t & (16) \\
 &= (t \otimes t)(\nu \otimes \nu)(\nu \otimes \nu)(t \otimes t)cc\delta t & (16) \\
 &= (t \otimes t)(\nu \otimes \nu)(\nu \otimes \nu)(t \otimes t)cc\delta & (5) \\
 &= (v \otimes v)cc\delta. & (5)
 \end{aligned}$$

An inverse for v is given by the morphism

$$v^{-1} = t\nu^{-1}\nu^{-1}t,$$

as may be seen in one direction by the calculation

$$\begin{aligned}
 v^{-1}v &= t\nu^{-1}\nu^{-1}tt\nu t \\
 &= tt\nu^{-1}\nu^{-1}\nu t t & (16) \\
 &= t\nu^{-1}\nu^{-1}\nu t & (7) \\
 &= tt \\
 &= t = 1_C. & (7)
 \end{aligned}$$

The other direction is similar.

Recall that the left $A \otimes A$ -, right A -coaction δ on P is defined by taking the diagonal of the commutative square:

$$\begin{array}{ccc} P & \xrightarrow{\delta_l} & A \otimes A \otimes P \\ \delta_r \downarrow & & \downarrow 1 \otimes 1 \otimes \delta_r \\ P \otimes A & \xrightarrow{\delta_l \otimes 1} & A \otimes A \otimes P \otimes A. \end{array}$$

We note that δ may be written as

We must show that θ is a left $A^{\otimes 3}$ -comodule isomorphism $P_l \rightarrow P_r$. That is, we must prove the commutativity of the square

$$\begin{array}{ccc} P_l & \xrightarrow{\gamma} & A^{\otimes 3} \otimes P_l \\ \theta \downarrow & & \downarrow 1 \otimes \theta \\ P_r & \xrightarrow{\gamma} & A^{\otimes 3} \otimes P_r \end{array}$$

where the left $A^{\otimes 3}$ -coactions on P_l and P_r were defined using δ (see §6.2).

The clockwise direction around the square is

where the last step (?) is given by the following calculation

The counter-clockwise direction is

Thus, θ is a left $A^{\otimes 3}$ -comodule morphism $P_l \longrightarrow P_r$. The inverse of θ then is a left $A^{\otimes 3}$ -comodule morphism $P_r \longrightarrow P_l$.

We now prove the properties (G1) through (G3) required of a quantum groupoid. The calculation

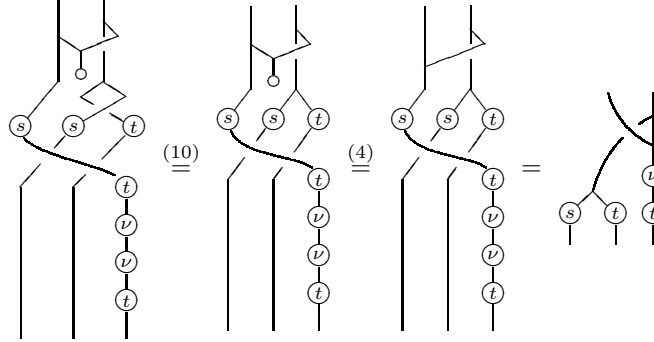
verifies (G1), and the following establishes (G2).

It remains to prove (G3), i.e., we must show that θ makes the following square

$$\begin{array}{ccc}
 P & \xrightarrow{\varsigma} & C^{\otimes 3} \xrightarrow{c_C, C^{\otimes 3} C} C^{\otimes 3} \\
 \theta \downarrow & & \downarrow 1 \otimes 1 \otimes v \\
 P & \xrightarrow{\varsigma} & C^{\otimes 3}
 \end{array}$$

commute.

The clockwise direction around the square is



for which the last step holds since

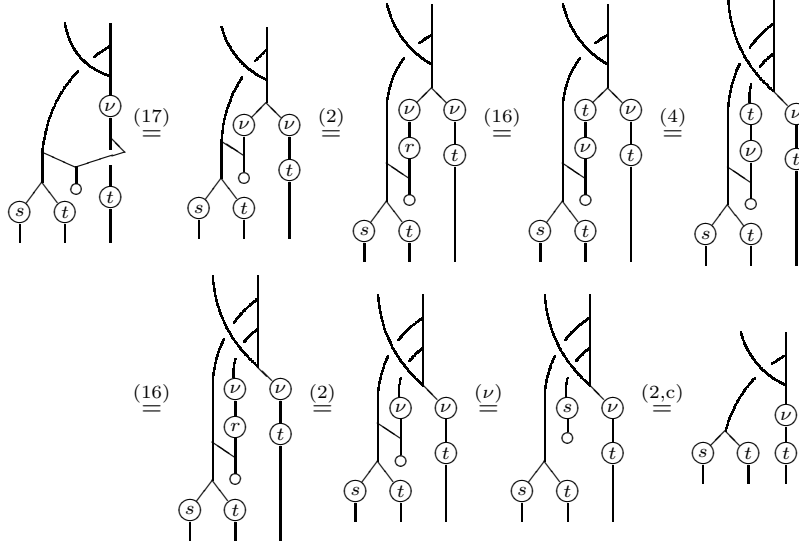
$$tvvts = tvvs \quad (8)$$

$$= tvr \quad (15)$$

$$= ttv \quad (16)$$

$$= tv. \quad (7)$$

The counter-clockwise direction is



therefore establishing the commutativity of the square.

Corollary 7.3. *Any Frobenius monoid in $\mathcal{Q}\mathcal{V}$ yields a quantum groupoid.*

By Proposition 5.1 every Frobenius monoid R in $\mathcal{Q}\mathcal{V}$ leads to a weak Hopf monoid with invertible antipode $R \otimes R$. Apply Proposition 7.1 to this weak Hopf monoid with invertible antipode to get a quantum groupoid.

APPENDIX A. STRING DIAGRAMS AND BASIC DEFINITIONS

In this appendix we give a quick introduction to string diagrams in a braided monoidal category $\mathcal{V} = (\mathcal{V}, \otimes, I, c)$ [13] and use these to define monoid, module,

comonoid, comodule, and separable Frobenius monoid in \mathcal{V} . The string calculus was shown to be rigorous in [12].

A.1. String diagrams. Suppose that $\mathcal{V} = (\mathcal{V}, \otimes, I, c)$ is a braided (strict) monoidal category. In a string diagram, objects label edges and morphisms label nodes. For example, if $f : A \otimes B \longrightarrow C \otimes D \otimes E$ is a morphism in \mathcal{V} it is represented as

$$f = \begin{array}{c} A \quad B \\ \diagdown \quad \diagup \\ \circ f \\ \diagup \quad \diagdown \\ C \quad D \quad E \end{array}$$

where this diagram is here meant to be read top-to-bottom. The identity morphism on an object will be represented as the object itself as in

$$A = \begin{array}{c} A \\ | \\ \cdot \end{array}.$$

A special case is the object $I \in \mathcal{V}$ which is represented as the empty edge.

If, in \mathcal{V} , there are morphisms $f : A \otimes B \longrightarrow C \otimes D \otimes E$ and $g : D \otimes E \otimes F \longrightarrow G \otimes H$ then they may be composed as

$$A \otimes B \otimes F \xrightarrow{f \otimes 1} C \otimes D \otimes E \otimes F \xrightarrow{1 \otimes g} C \otimes G \otimes H$$

which may be represented as vertical concatenation

$$(1 \otimes g)(f \otimes 1) = \begin{array}{c} \diagdown \quad \diagup \\ \circ f \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \circ g \\ \diagup \quad \diagdown \end{array}$$

(where we have left off the objects). The tensor product of morphisms, say

$$\begin{array}{c} \diagdown \quad \diagup \\ \circ f \\ \diagup \quad \diagdown \end{array} \quad \text{and} \quad \begin{array}{c} \diagdown \quad \diagup \\ \circ g \\ \diagup \quad \diagdown \end{array},$$

is represented as horizontal juxtaposition

$$f \otimes g = \begin{array}{c} \diagdown \quad \diagup \\ \circ f \\ \diagup \quad \diagdown \end{array} \quad \begin{array}{c} \diagdown \quad \diagup \\ \circ g \\ \diagup \quad \diagdown \end{array}$$

(again leaving off the objects).

The braiding $c_{A,B} : A \otimes B \longrightarrow B \otimes A$ is represented as a left-over-right crossing. The inverse braiding is then represented as a right-over-left crossing.

$$c_{A,B} = \begin{array}{c} A \quad B \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ B \quad A \end{array} \quad c_{A,B}^{-1} = \begin{array}{c} B \quad A \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ A \quad B \end{array}$$

Suppose $A \in \mathcal{V}$ has a left dual A^* , which we denote by $A^* \dashv A$ and say that A^* is the left adjoint of A (it is an adjunction if we were to view \mathcal{V} as a one object

bicategory). The evaluation and coevaluation morphisms $e_A : A^* \otimes A \longrightarrow I$ and $n_A : I \longrightarrow A \otimes A^*$ are represented as

$$e_A = \begin{array}{c} A^* \quad A \\ \cup \end{array} \quad \text{and} \quad n_A = \begin{array}{c} \cap \\ A \quad A^* \end{array}.$$

The triangle equalities become

$$\begin{array}{c} A \\ | \\ \cap \\ A^* \quad A \\ | \\ A \end{array} = \begin{array}{c} A \\ | \end{array} \quad \text{and} \quad \begin{array}{c} A^* \\ | \\ \cap \\ A \quad A^* \\ | \\ A^* \end{array} = \begin{array}{c} A^* \\ | \end{array}$$

In what follows, in order to simplify the string diagrams, the nodes will be omitted from certain morphisms (e.g., multiplication and comultiplication morphisms) or simplified (e.g., unit and counit morphisms).

A.2. Monoids and modules. A *monoid* $A = (A, \mu, \eta)$ in \mathcal{V} is an object $A \in \mathcal{V}$ equipped with morphisms

$$\mu = \begin{array}{c} \diagup \quad \diagdown \\ | \end{array} : A \otimes A \longrightarrow A \quad \text{and} \quad \eta = \begin{array}{c} \circ \\ | \end{array} : I \longrightarrow A,$$

called the *multiplication* and *unit* of the monoid respectively, satisfying

$$(m) \quad \begin{array}{c} \diagup \quad \diagdown \\ | \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ | \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ | \end{array} \quad \text{and} \quad \begin{array}{c} \circ \\ | \end{array} = \begin{array}{c} | \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ | \end{array} \circ.$$

If A, B are monoids, a *monoid morphism* $f : A \longrightarrow B$ is a morphism in \mathcal{V} satisfying

$$\begin{array}{c} A \quad A \\ \diagup \quad \diagdown \\ | \\ \circ \\ | \\ B \end{array} = \begin{array}{c} A \quad A \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ | \quad | \\ f \quad f \\ | \quad | \\ B \end{array} \quad \text{and} \quad \begin{array}{c} A \\ \circ \\ | \\ \circ \\ | \\ B \end{array} = \begin{array}{c} \circ \\ | \\ B \end{array}.$$

Monoids make sense in any monoidal category, however, in order that the tensor product $A \otimes B$ of monoids $A, B \in \mathcal{V}$ is again a monoid there must be a “switch” morphism $c_{A,B} : A \otimes B \longrightarrow B \otimes A$ in \mathcal{V} given by, say, a braiding. In this case $A \otimes B$ becomes a monoid in \mathcal{V} via

$$\mu = \begin{array}{c} \diagup \quad \diagdown \\ | \end{array} \quad \text{and} \quad \eta = \begin{array}{c} \circ \quad \circ \\ | \end{array}.$$

Suppose that A is a monoid in \mathcal{V} . A *right A -module* in \mathcal{V} is an object $M \in \mathcal{V}$ equipped with a morphism

$$\mu = \begin{array}{c} M \quad A \\ \diagup \quad \diagdown \\ | \end{array} : M \otimes A \longrightarrow M$$

called the *action of A on M* satisfying

$$(m) \quad \begin{array}{c} M \quad A \quad A \\ \diagdown \quad \diagup \\ \text{---} \\ M \end{array} = \begin{array}{c} M \quad A \quad A \\ \diagdown \quad \diagup \\ \text{---} \\ M \end{array} \quad \text{and} \quad \begin{array}{c} M \\ \diagdown \quad \diagup \\ \text{---} \\ M \end{array} \circ_A = \begin{array}{c} M \\ \text{---} \\ M \end{array}.$$

Notice that we use the same label “(m)” as the monoid axioms (and “(c)” below for the comodule axioms). This should not cause any confusion as the labelling of strings disambiguates a multiplication and an action; however, the labelling will usually be left off.

If M, N are modules, a *module morphism* $f : M \rightarrow N$ is a morphism in \mathcal{V} satisfying

$$\begin{array}{c} M \quad A \\ \diagdown \quad \diagup \\ \text{---} \\ N \end{array} \circ_f = \begin{array}{c} M \quad A \\ \diagdown \quad \diagup \\ \text{---} \\ N \end{array}.$$

A.3. Comonoids and comodules. Comonoids and comodules are dual to monoids and modules. Explicitly, a *comonoid* $C = (C, \delta, \epsilon)$ in \mathcal{V} is an object $C \in \mathcal{V}$ equipped with morphisms

$$\delta = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} : A \rightarrow A \otimes A \quad \text{and} \quad \epsilon = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \circ \end{array} : A \rightarrow I,$$

called the *comultiplication* and *counit* of the comonoid respectively, satisfying

$$(c) \quad \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \circ \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \circ \end{array} = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \circ \end{array}.$$

If C, D are comonoids, a *comonoid morphism* $f : C \rightarrow D$ is a morphism in \mathcal{V} satisfying

$$\begin{array}{c} A \\ \diagdown \quad \diagup \\ \text{---} \\ B \quad B \end{array} \circ_f = \begin{array}{c} A \\ \diagdown \quad \diagup \\ \text{---} \\ B \quad B \end{array} \quad \text{and} \quad \begin{array}{c} A \\ \diagdown \quad \diagup \\ \text{---} \\ B \quad B \end{array} \circ_f = \begin{array}{c} A \\ \text{---} \\ B \quad B \end{array}.$$

Similarly here, \mathcal{V} must contain a switch morphism $c_{C,D} : C \otimes D \rightarrow D \otimes C$ in order that the tensor product $C \otimes D$ of comonoids $C, D \in \mathcal{V}$ is again a comonoid. In this case the comultiplication and counit are given by

$$\delta = \begin{array}{c} \text{---} \quad \text{---} \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \text{---} \end{array} \quad \text{and} \quad \epsilon = \begin{array}{c} \text{---} \quad \text{---} \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \circ \quad \circ \end{array}.$$

Suppose that C is a comonoid in \mathcal{V} . A *right C -comodule* in \mathcal{V} is an object $M \in \mathcal{V}$ equipped with a morphism

$$\gamma = \begin{array}{c} M \\ \diagdown \quad \diagup \\ \text{---} \\ M \quad C \end{array} : M \rightarrow M \otimes C$$

called the *coaction of A on M* satisfying

$$(c) \quad \begin{array}{c} M \\ \diagdown \quad \diagup \\ \text{---} \\ M \quad C \quad C \end{array} = \begin{array}{c} M \\ \diagdown \quad \diagup \\ \text{---} \\ M \quad C \quad C \end{array} \quad \text{and} \quad \begin{array}{c} M \\ \diagdown \quad \diagup \\ \text{---} \\ M \quad C \end{array} \circ_C = \begin{array}{c} M \\ \text{---} \\ M \end{array}.$$

If M, N are C -comodules, a *comodule morphism* $f : M \longrightarrow N$ is a morphism in \mathcal{V} satisfying

$$\begin{array}{c} M \\ | \\ \textcircled{f} \\ / \quad \backslash \\ N \quad C \end{array} = \begin{array}{c} M \\ | \\ \textcircled{f} \\ | \quad \backslash \\ N \quad C \end{array}.$$

In this paper we also make use of C -bicomodules. Suppose that M is both a left C -comodule and a right C -comodule with coactions

$$\gamma_l : M \longrightarrow C \otimes M$$

$$\gamma_r : M \longrightarrow M \otimes C.$$

If the square

$$\begin{array}{ccc} M & \xrightarrow{\gamma_l} & C \otimes M \\ \gamma_r \downarrow & & \downarrow 1 \otimes \gamma_r \\ M \otimes C & \xrightarrow{\gamma_l \otimes 1} & C \otimes M \otimes C \end{array}$$

commutes, meaning

$$\begin{array}{c} C \\ | \\ / \quad \backslash \\ M \quad C \quad M \end{array} = \begin{array}{c} C \\ | \\ / \quad \backslash \\ M \quad C \quad M \end{array}$$

in string diagrams, then M is called a C -bicomodule. The diagonal of the square will be denoted by

$$\gamma : M \longrightarrow C \otimes M \otimes C.$$

A.4. Frobenius monoids. A *Frobenius monoid* R in \mathcal{V} is both a monoid and a comonoid in \mathcal{V} which additionally satisfies the ‘‘Frobenius condition’’:

$$\begin{array}{ccc} R \otimes R & \xrightarrow{\delta \otimes 1} & R \otimes R \otimes R \\ 1 \otimes \delta \downarrow & & \downarrow 1 \otimes \mu \\ R \otimes R \otimes R & \xrightarrow{\mu \otimes 1} & R \otimes R. \end{array}$$

In strings the Frobenius condition is displayed as

$$\begin{array}{c} | \\ | \\ | \\ | \end{array} = \begin{array}{c} | \\ | \\ | \\ | \end{array}.$$

We will now review some basic facts about Frobenius monoids.

Lemma A.1. $(1 \otimes \mu)(\delta \otimes 1) = \delta \mu = (\mu \otimes 1)(1 \otimes \delta) : R \otimes R \longrightarrow R \otimes R.$

Proof. The left-hand identity is proved by the following string calculation.

$$\begin{array}{c} | \\ | \\ | \\ | \end{array} = \begin{array}{c} | \\ | \\ | \\ | \end{array} \circ = \begin{array}{c} | \\ | \\ | \\ | \end{array} \circ = \begin{array}{c} | \\ | \\ | \\ | \end{array} \circ = \begin{array}{c} | \\ | \\ | \\ | \end{array} \circ = \begin{array}{c} | \\ | \\ | \\ | \end{array}$$

The right-hand identity follows from a similar calculation. □

$$\begin{aligned}\rho &= (I \xrightarrow{\eta} R \xrightarrow{\delta} R \otimes R) = \text{fork} \\ \sigma &= (R \otimes R \xrightarrow{\mu} R \xrightarrow{\epsilon} I) = \text{unfork}.\end{aligned}$$

Proof. One of the triangle identities is given as

Proof. Given $f : R \longrightarrow S$ define $f^{-1} : S \longrightarrow R$ by

$$f^{-1} = S \xrightarrow{\rho \otimes 1} R \otimes R \otimes S \xrightarrow{1 \otimes f \otimes 1} R \otimes S \otimes S \xrightarrow{1 \otimes \sigma} R = \text{diagram}$$

[illegible]
$$S \xrightarrow{1 \otimes \rho} S \otimes R \otimes R \xrightarrow{1 \otimes f \otimes 1} S \otimes S \otimes R \xrightarrow{\sigma \otimes 1} R = \text{diagram}.$$

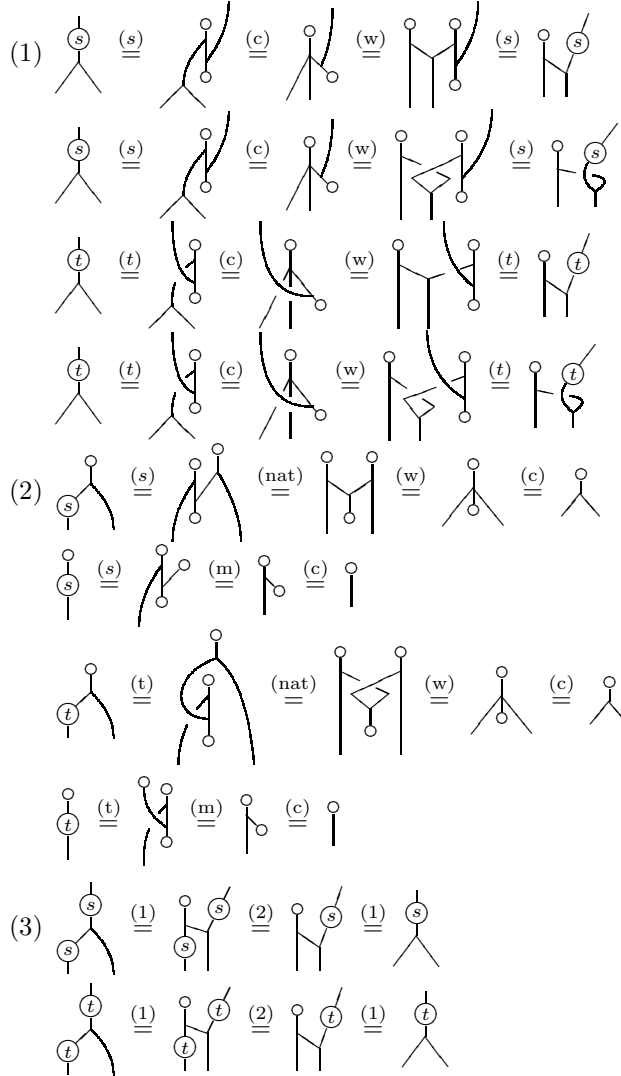
Corollary A.4. *For any morphism of Frobenius monoids $f : R \rightarrow S$ we have*

Definition A.5. A Frobenius monoid R is said to be *separable* if and only if $\mu\delta = 1$, i.e.,

$$\begin{array}{c} \diagup \\ | \\ \diagdown \end{array} = \begin{array}{c} | \end{array}.$$

APPENDIX B. PROOFS OF THE PROPERTIES OF s , t , AND r

As we have noted in §2, $s : A \longrightarrow A$ is invariant under rotation by π , $t : A \longrightarrow A$ is invariant under horizontal reflection, and r is t rotated by π . This reduces the number of proofs we present as the others are derivable.



(4)

(5)

(6)

(7)

$$(8) \begin{array}{c} \text{Diagram 1} \end{array} \stackrel{(s)}{=} \begin{array}{c} \text{Diagram 2} \end{array} \stackrel{(2)}{=} \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(s)}{=} \begin{array}{c} \text{Diagram 4} \end{array}$$

$$\begin{array}{c} \text{Diagram 5} \end{array} \stackrel{(t)}{=} \begin{array}{c} \text{Diagram 6} \end{array} \stackrel{(2)}{=} \begin{array}{c} \text{Diagram 7} \end{array} \stackrel{(t)}{=} \begin{array}{c} \text{Diagram 8} \end{array}$$

$$(9) \begin{array}{c} \text{Diagram 1} \end{array} \stackrel{(3)}{=} \begin{array}{c} \text{Diagram 2} \end{array} \stackrel{(8)}{=} \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(3)}{=} \begin{array}{c} \text{Diagram 4} \end{array}$$

$$\begin{array}{c} \text{Diagram 5} \end{array} \stackrel{(3)}{=} \begin{array}{c} \text{Diagram 6} \end{array} \stackrel{(8)}{=} \begin{array}{c} \text{Diagram 7} \end{array} \stackrel{(3)}{=} \begin{array}{c} \text{Diagram 8} \end{array}$$

$$(10) \begin{array}{c} \text{Diagram 1} \end{array} \stackrel{(s,t)}{=} \begin{array}{c} \text{Diagram 2} \end{array} \stackrel{(\text{nat})}{=} \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(\text{v})}{=} \begin{array}{c} \text{Diagram 4} \end{array} \stackrel{(s,t)}{=} \begin{array}{c} \text{Diagram 5} \end{array}$$

$$(11) \begin{array}{c} \text{Diagram 1} \end{array} \stackrel{(\text{m})}{=} \begin{array}{c} \text{Diagram 2} \end{array} \stackrel{(4)}{=} \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(\text{m})}{=} \begin{array}{c} \text{Diagram 4} \end{array} \stackrel{(2)}{=} \begin{array}{c} \text{Diagram 5} \end{array} \stackrel{(2)}{=} \begin{array}{c} \text{Diagram 6} \end{array} \stackrel{(4)}{=} \begin{array}{c} \text{Diagram 7} \end{array}$$

$$(12) \begin{array}{c} \text{Diagram 1} \end{array} \stackrel{(r,s)}{=} \begin{array}{c} \text{Diagram 2} \end{array} \stackrel{(\text{nat})}{=} \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(\text{v})}{=} \begin{array}{c} \text{Diagram 4} \end{array} \stackrel{(\text{c})}{=} \begin{array}{c} \text{Diagram 5} \end{array} \stackrel{(s)}{=} \begin{array}{c} \text{Diagram 6} \end{array}$$

$$(13) \begin{array}{c} \text{Diagram 1} \end{array} \stackrel{(t,r)}{=} \begin{array}{c} \text{Diagram 2} \end{array} \stackrel{(\text{nat})}{=} \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(\text{v})}{=} \begin{array}{c} \text{Diagram 4} \end{array} \stackrel{(t,r)}{=} \begin{array}{c} \text{Diagram 5} \end{array}$$

$$(14) \begin{array}{c} \text{Diagram 1} \end{array} \stackrel{(3)}{=} \begin{array}{c} \text{Diagram 2} \end{array} \stackrel{(12)}{=} \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(3)}{=} \begin{array}{c} \text{Diagram 4} \end{array}$$

$$\begin{array}{c} \text{Diagram 5} \end{array} \stackrel{(3)}{=} \begin{array}{c} \text{Diagram 6} \end{array} \stackrel{(12)}{=} \begin{array}{c} \text{Diagram 7} \end{array} \stackrel{(3)}{=} \begin{array}{c} \text{Diagram 8} \end{array}$$

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